Longitudinal Dynamics (OCPA 2025)

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Contents

A lot of references including below (but not all):

- Wolski - book
- A. Chao - book
- S.Y. Lee - book
- R. H. Siemann, Bunched beam diagnostics
- ...

Focus on circular machine
If any error, pls let me know.

1 Not so prerequisite

Configuration Space: 3D
Phase Space: 6D

Transverse: Dynamic Aperture (ring)
Longitudinal: Momentum Acceptance (ring)

Dynamic Coordinates

- x
- $p_x \approx \frac{dx}{ds}$
- $z = \frac{s}{\beta_0} - ct$. Different definition in different code. Here we follow Wolski (MADX?). In the note, maybe we don't keep same definition. (Sorry!)
- Instead of $p_z \approx \frac{dz}{ds}$, $p_z = \delta = \frac{\Delta E}{P_0 c}$. Since δ nearly constant along the ring.

Drift (Wolski)

$$H_{\text{drift}}(x, p_x, y, p_y, z, \delta; s) = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \tag{1}$$

$$\begin{cases} \frac{dp_x}{ds} = 0, \\ \frac{dx}{ds} = \frac{\partial H}{\partial p_x} = \frac{p_x}{D}, \\ \frac{d\delta}{ds} = 0, \\ \frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0}\left(1 - \frac{1}{D}\right) - \frac{\delta}{D}, \end{cases} \Rightarrow \begin{cases} x_1 = x_0 + \frac{p_{x0}}{D} \cdot L, \\ y_1 = y_0 + \frac{p_{y0}}{D} \cdot L, \\ z_1 = z_0 + \frac{L}{\beta_0}\left(1 - \frac{1}{D}\right) - \frac{\delta}{D} \cdot L, \end{cases} \tag{2}$$

where $D \equiv \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}$

Linear Approximation of Drift Map

$$H \approx \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \Rightarrow \begin{cases} x_1 = x_0 + L p_{x0}, \\ y_1 = y_0 + L p_{y0}, \\ z_1 = z_0 + \frac{L}{\beta_0^2\gamma_0^2} \cdot \delta_0 \end{cases} \tag{3}$$

Quadrupole

With $K_1 = \frac{q}{p_0} \cdot \frac{\partial B_y}{\partial x}$,

$$H_Q = H_{\text{drift}} + \frac{K_1}{2}(x^2 - y^2), \tag{4}$$

$$H_Q \approx \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \frac{K_1}{2}(x^2 - y^2). \tag{5}$$

Linear Map:

$$M_{\text{Quad}} = \begin{pmatrix} \cos & \frac{\sin}{\sqrt{K_1}} & & & & \\ -\sqrt{K_1}\sin & \cos & & & & \\ & & \cosh & \frac{\sinh}{\sqrt{K_1}} & & \\ & & \sqrt{K_1}\sinh & \cosh & & \\ & & & & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ & & & & 0 & 1 \end{pmatrix}. \tag{6}$$

The parameter of sin, cos, sinh, cosh is $\sqrt{K_1}L$

It is well known that focus strength will be reduced for higher energy particle. This is not linear effect. If only taking 2nd order approximation of p_x/p_y, but kepp exact δ terms,

$$H_Q = \frac{p_x^2}{2D_\delta} + \frac{p_y^2}{2D_\delta} + \frac{\delta}{\beta_0} - D_\delta + \frac{K_1}{2}(x^2 - y^2), \tag{7}$$

where $D_\delta = \sqrt{1 + \frac{2\delta}{\beta_0} + \delta^2}$.
The transverse transfer matrix with chromatic effect,

$$\begin{pmatrix} \cos & \frac{\sin}{\sqrt{K_1' D_\delta}} & & \\ -\sqrt{K_1'} D_\delta \sin & \cos & & \\ & & \cosh & \frac{\sinh}{\sqrt{K_1'} D_\delta} \\ & & \sqrt{K_1'} D_\delta \sinh & \cosh \end{pmatrix} \tag{8}$$

where $K_1' = \frac{K_1}{D_\delta}$ and the parameter of $\sin, \cos, \sinh, \cosh$ is $\sqrt{K_1'} L$.

RF Cavity

A Pillbox Cavity, TM010

$$E_z = E_0 \cdot J_0(kr) \cdot \sin(\omega t + \phi_0) \tag{9}$$

$$B_\theta = \frac{E_0}{c} \cdot J_1(kr) \cdot \cos(\omega t + \phi_0) \tag{10}$$

where $\omega \approx 2.40483 \cdot \frac{c}{a}$ and $k = \frac{\omega}{c} = \frac{2.40483}{a}$. a is radius. Reminder: $J_0(2.40483) = 0$.
The vector potential is

$$\vec{A} = \left(0,\ 0,\ \frac{E_0}{\omega} \cdot J_0(kr) \cdot \cos(\omega t + \phi_0)\right) \tag{11}$$

With $z = \frac{s}{\beta_0} - ct$ and $\omega t = \frac{ks}{\beta_0} - kz$, Hamiltonian of Cavity is

$$H_{\text{cav}} = H_{\text{drift}} - \frac{q}{P_0} \cdot A_s, \quad r = \sqrt{x^2 + y^2} \tag{12}$$

H_{cav} depend on s!
For a simplified model, take average of H_{cav} over cavity length,

$$\begin{aligned} \langle H \rangle_{\text{cav}} &= \frac{1}{L} \int_{-L/2}^{+L/2} H_{\text{cav}}\, \mathrm{d}s \\ &= H_{\text{drift}} - \frac{1}{L} \cdot \frac{q}{P_0} \cdot \frac{2E_0\beta_0}{c \cdot k^2} \cdot \sin\left(\frac{kL}{2\beta_0}\right) \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\ &= H_{\text{drift}} - \frac{qE_0}{P_0 ck} \cdot T \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \end{aligned} \tag{13}$$

where transit time factor $T \equiv \frac{\sin\left(\frac{kL}{2\beta_0}\right)}{\frac{kL}{2\beta_0}}$

Kick approximation.
$L \to 0$, but $V_{\text{rf}} \equiv E_0 L T$ keep constant.

$$\begin{aligned}\langle H\rangle_{\text{cav}} L &= -\frac{q}{P_0 c k} \cdot E_0 \cdot L \cdot T \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\ &= -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\ &\approx -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \cos(\phi_0 - kz)\end{aligned} \tag{14}$$

Approximation of small transverse amplitude is used in above equation.

2nd order approximation.

$$\langle H\rangle_{\text{cav}} L \approx = -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \left[\cos(\phi_0) + \sin(\phi_0) \cdot kz - \frac{\cos(\phi_0) \cdot k^2 z^2}{2}\right] \tag{15}$$

The change of δ through the RF cavity is

$$\Delta\delta = -\frac{\partial(\langle H\rangle_{\text{cav}} L)}{\partial z} = \frac{q}{P_0 c} \cdot V_{\text{rf}} \cdot \sin(\phi_0) - \frac{q}{P_0 c} \cdot V_{\text{rf}} k \cos(\phi_0) \cdot kz \tag{16}$$

We could also get the map with finite L and truncate $\langle H\rangle$ with 2nd order approximation. See Wolski's book.

Change of reference momentum.
The reference momentum after RF cavity is not changed before. We could also change the reference momentum $\mathrm{P}_0 \to \mathrm{P}_1$. (CEPC). In this case, the conjugate momentum should be changed.
Recall

$$\begin{aligned} p_{x,0} &= \frac{\beta_x \gamma m c}{\mathrm{P}_0}, \\ \delta_0 &= \frac{\mathrm{E}}{c\mathrm{P}_0} - \frac{1}{\beta_0}\end{aligned} \tag{17}$$

We have

$$\begin{aligned} p_{x0} \cdot \mathrm{P}_0 &= p_{x1} \cdot \mathrm{P}_1 \\ p_{y0} \cdot \mathrm{P}_0 &= p_{y1} \cdot \mathrm{P}_1 \\ \delta_1 &= \frac{\mathrm{P}_0}{\mathrm{P}_1}\delta_0 - \frac{1}{\beta_1} + \frac{\gamma_0}{\beta_1\gamma_1}\end{aligned} \tag{18}$$

Matrix representation:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_1 = \begin{pmatrix} 1 & & & & & \\ & \frac{\mathrm{P}_0}{\mathrm{P}_1} & & & & \\ & & 1 & & & \\ & & & \frac{\mathrm{P}_0}{\mathrm{P}_1} & & \\ & & & & 1 & \\ & & & & & \frac{\mathrm{P}_0}{\mathrm{P}_1} \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_0 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\beta_1} + \frac{\gamma_0}{\beta_1\gamma_1} \end{pmatrix} \tag{19}$$

For example, if ref momentum at s is determined, we could recalc the phase space coordinates with the local ref momentum. This could help "reduce" the number of (p_x, p_y, δ). Small number is often preferred especially approximation is used.

The above transformation is non-symplectic.
How the emittance and Twiss function change with the reference momentum?

$$\Sigma = \langle \vec{x} \cdot \vec{x}^{\mathrm{T}} \rangle = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \tag{20}$$

With

$$\vec{x}_1 = R\vec{x}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{P}_0/\mathrm{P}_1 \end{pmatrix} \vec{x}_0, \quad \Sigma_1 = R \cdot \Sigma_0 \cdot R^{\mathrm{T}} \tag{21}$$

$$\begin{pmatrix} \beta_{x1} & -\alpha_{x1} \\ -\alpha_{x1} & \gamma_{x1} \end{pmatrix} \varepsilon_{x1} = \begin{pmatrix} \beta_{x0} & -\frac{\mathrm{P}_0}{\mathrm{P}_1}\alpha_{x0} \\ -\frac{\mathrm{P}_0}{\mathrm{P}_1}\alpha_{x0} & \left(\frac{\mathrm{P}_0}{\mathrm{P}_1}\right)^2 \gamma_{x0} \end{pmatrix} \varepsilon_{x0} \tag{22}$$

$$\varepsilon_{x1} = \frac{\mathrm{P}_0}{\mathrm{P}_1} \cdot \varepsilon_{x0} \implies \beta_1\gamma_1\varepsilon_{x1} = \beta_0\gamma_0\varepsilon_{x0} \tag{23}$$

$$\implies \begin{cases} \beta_{x1} = \frac{\mathrm{P}_1}{\mathrm{P}_0} \cdot \beta_{x0} \\ \alpha_{x1} = \alpha_{x0} \\ \gamma_{x1} = \frac{\mathrm{P}_0}{\mathrm{P}_1} \cdot \gamma_{x0} \end{cases} \tag{24}$$

example: CEPC Twiss function

Dipole

$h = \frac{1}{\rho}$ is the curvature of reference trajectory.
Normalized field strength $k_0 = \frac{q}{\mathrm{P}_0} B_0$
Sacled vector potential: $\vec{a} = \frac{q}{P_0}\vec{A} = \left(0, 0, -k_0 x + \frac{k_0 h x^2}{2(1+hx)}\right)$
Approximated Hamiltonian (2nd order):

$$H_2 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \left(k_0 - h\left(1 + \frac{\delta}{\beta_0}\right)\right) x + \frac{k_0 h x^2}{2}. \tag{25}$$

With $h = k_0$ in most cases, the transfer matrix:

$$\begin{pmatrix} \cos\left(\frac{L}{\rho}\right) & \rho\sin\left(\frac{L}{\rho}\right) & & & & \frac{\rho}{\beta_0}\left[1 - \cos\left(\frac{L}{\rho}\right)\right] \\ -\sin\left(\frac{L}{\rho}\right)/\rho & \cos\left(\frac{L}{\rho}\right) & & & & \frac{1}{\beta_0}\sin\left(\frac{L}{\rho}\right) \\ & & 1 & L & & \\ & & 0 & 1 & & \\ -\frac{1}{\beta_0}\sin\left(\frac{L}{\rho}\right) & -\frac{\rho}{\beta_0}\left[1 - \cos\left(\frac{L}{\rho}\right)\right] & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{1}{\beta_0^2}\left(L - \rho\sin\left(\frac{L}{\rho}\right)\right) \\ & & & & 0 & 1 \end{pmatrix} \tag{26}$$

For $h \neq k_0$ and with k_1, see Wolski.

Path Length

Trajectory:

$$\vec{r}(s) = \vec{r}_0(s) + x\hat{x} + y\hat{y} \tag{27}$$

Length of trajectory:

$$\begin{aligned}\oint |d\vec{r}(s)| &= \oint \left|\hat{s}ds + x'\hat{x}ds + \frac{x}{\rho}ds\hat{s} + y'\hat{y}ds\right| \\ &= \oint ds\sqrt{(1+\frac{x}{\rho})^2 + x'^2 + y'^2} \approx C + \oint ds\frac{x}{\rho}\end{aligned} \tag{28}$$

The path length difference of closed orbit

$$\Delta C = \oint \frac{x_{\text{COD}}(s)}{\rho(s)}ds$$

Considering the cod induced by dispersion:

$$\Delta C = \delta \oint \frac{D(s)}{\rho(s)}ds \tag{29}$$

$$\frac{\Delta C}{C} = \alpha_c \delta, \qquad \text{or} \qquad \alpha_c = \frac{1}{\delta}\frac{\Delta C}{C} \tag{30}$$

Momentum compaction factor:

$$\alpha_c = \frac{1}{C}\oint \frac{D(s)}{\rho(s)}ds \tag{31}$$

Consider a storage ring which contains a horizontal kick with angle θ at s_0.
COD caused by this kick induces a change of orbital circumference:

$$\Delta C = \oint ds \frac{x_{COD}}{\rho(s)} = \theta \cdot D(s_0). \tag{32}$$

D is the dispersion.
With RF systems, ΔC has to be compensated by an adjustment of energy

$$\Delta\delta = -\frac{1}{\alpha_c}\frac{\Delta C}{C} = -\frac{1}{\alpha_c}\frac{\theta \cdot D(s_0)}{C} \tag{33}$$

A more accurate COD formula:

$$x_{\text{COD}}(s) = \theta\frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos(\pi\nu - |\psi(s) - \psi(s_0)|) - \theta\frac{D(s_0)\cdot D(s)}{\alpha_c \cdot C} \tag{34}$$

Symmetric DBA (ref S.Y. Lee)

$$\begin{aligned} \alpha_c &= \frac{\rho}{L_m}\left[\theta - \sin\theta + \frac{D_0 \sin\theta}{\rho} + D_0'(1-\cos\theta)\right] \\ &= \frac{\rho\theta^3}{L_m}\left(\frac{1}{6} + \frac{D_0}{L\theta} + \frac{D_0'}{2\theta}\right) + \dots \end{aligned} \tag{35}$$

where

- D_0, D_0': entrance of dipole
- L_m: half length of DBA
- ρ: bending radius, $\theta = L/\rho$, L: length of dipole

With $D_0 = 0$, $D_0' = 0$,

$$\alpha_c = \frac{\rho\theta^3}{6L_m} = \frac{\rho\theta^3}{6C\theta/(2\pi)} = \frac{\rho\theta^2}{6R} \tag{36}$$

α_c could be negative.

2 Basic Longitudinal dynamics

The potentail well term of Hamiltonian contributed by Cavity,

$$H_{\text{cav}} = -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \cos(\phi_0 - kz) \tag{37}$$

We ignore the length L in the following.
Sine we follow Wolski define

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} = \frac{E - E_0}{P_0 c} \tag{38}$$

And the momentum deviation

$$\delta_p = \frac{P - P_0}{P_0} \tag{39}$$

we have

$$\begin{aligned} \left.\frac{d\delta}{d\delta_p}\right|_{\delta_p=0} &= \beta_0 \\ \left.\frac{d}{d\delta_p}\right|_{\delta_p=0} &= \beta_0 \left.\frac{d}{d\delta}\right|_{\delta=0} \end{aligned} \tag{40}$$

$\Delta\delta$ through RF cavity

$$\Delta\delta = -\frac{\partial H}{\partial z} = \frac{q}{P_0 c} \cdot V_{rf} \sin(\phi_0 - \omega_{rf} \cdot \frac{z}{c}) \tag{41}$$

Considering energy loss U_0 due to SR, $\Delta\delta$ in one turn,

$$\Delta\delta = \frac{q}{P_0 c} \cdot V_{rf} \sin(\phi_0 - \omega_{rf} \cdot \frac{z}{c}) - \frac{U_0}{P_0 c} \tag{42}$$

Take the average in the whole ring

$$\frac{d\delta}{ds} = \frac{qV_{rf}}{cP_0C_0} \cdot \sin(\phi_0 - \omega_{rf} \cdot \frac{z}{c}) - \frac{U_0}{cP_0C_0} \tag{43}$$

With momentum compaction factor

$$\alpha_p = \frac{1}{C_0}\left.\frac{dC}{d\delta_p}\right|_{\delta_p=0} = \frac{1}{C_0}\int_0^{C_0}\frac{\eta_x}{\rho}ds \tag{44}$$

Phase slip factor

$$\begin{aligned} T &= \frac{C}{\beta c} \Rightarrow \\ \frac{1}{T}\frac{dT}{d\delta_p} &= \frac{1}{C}\frac{dC}{d\delta_p} - \frac{1}{\beta}\frac{d\beta}{d\delta_p} \Rightarrow \\ \eta_p &= \frac{1}{T_0}\left.\frac{dT}{d\delta_p}\right|_{\delta_p=0} = \alpha_p - \frac{1}{\gamma_0^2} \end{aligned} \tag{45}$$

α_p is the property of lattice.
Change design momentum of lattice, magnet strength is scaled and α_p will keep unchanged.
γ_0 the design beam energy
Transition energy:

$$\eta_p = 0 \quad \Rightarrow \quad \gamma_T = \frac{1}{\sqrt{\alpha_p}} \tag{46}$$

With $z = \frac{s}{\beta_0} - ct$,
at $s = s_0$ and $t = t_0$,
after one turn back at $s = s_0$, $t = t_0 + T$ (no dispersion at s_0

$$\Delta z = \frac{C_0}{\beta_0} - cT \tag{47}$$

We have

$$R_{56} = \frac{\partial z}{\partial \delta} = -c\frac{dT}{d\delta} = -c\frac{1}{\beta_0}\frac{dT}{d\delta_p} = -\frac{C}{\beta_0^2}\eta_p \tag{48}$$

Average Δz over the ring,

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{T}{\beta_0 T_0} = -\frac{1}{\beta_0}\eta_p\delta_p = -\frac{1}{\beta_0^2}\eta_p\delta \tag{49}$$

Combine $\frac{d\delta}{ds}$ and $\frac{dz}{ds}$, we get

$$\begin{aligned}\frac{d^2z}{ds^2} &= -\frac{1}{\beta_0^2}\cdot\eta_p\cdot\frac{d\delta}{ds} \\ &= -\frac{1}{\beta_0^2}\cdot\eta_p\cdot\frac{qV_{rf}}{cP_0C_0}\sin(\phi_0-\omega_{rf}\frac{z}{c})+\frac{1}{\beta_0^2}\cdot\eta_p\cdot\frac{U_0}{cP_0C_0}\end{aligned} \tag{50}$$

If $\phi_0=\phi_s$, $\sin\phi_s=\frac{U_0}{qV_{rf}}$, take the first order approximation of z in RHS,

$$\frac{d^2z}{ds^2}+\frac{1}{\beta_0^2}\frac{qV_{rf}}{cP_0C_0}\frac{\omega_{rf}}{c}\eta_p\cos\phi_s\cdot z=0 \tag{51}$$

⇒ longitudinal tune

$$\begin{aligned}\nu_s = \frac{C_0}{2\pi}\sqrt{-\frac{1}{\beta_0^2}\frac{qV_{rf}}{cP_0C_0}\frac{\omega_{rf}}{c}\eta_p\cos\phi_s} &= \sqrt{-\frac{1}{2\pi}\frac{1}{\beta_0}\frac{qV_{rf}}{cP_0}\cdot h\cdot\eta_p\cos\phi_s} \\ &= \sqrt{-\frac{1}{2\pi}\cdot\frac{1}{\beta_0^2}\frac{qV_{rf}}{E_0}\cdot h\cdot\eta_p\cos\phi_s}\end{aligned} \tag{52}$$

where $\omega_{rf}=h\omega_0$ is used.

Hamiltonian of Longitudinal Oscillation

With equation of $\frac{dz}{ds}$ and $\frac{d\delta}{ds}$, the Hamiltonian could be obtained

$$H(z,\delta;s)=\frac{qV_{rf}}{w_{rf}P_0C_0}\left[-\cos(\phi_s-\frac{w_{rf}z}{c})+\sin\phi_s(\frac{w_{rf}z}{c})\right]-\frac{1}{2}\frac{1}{\beta_0^2}\eta_p\cdot\delta^2 \tag{53}$$

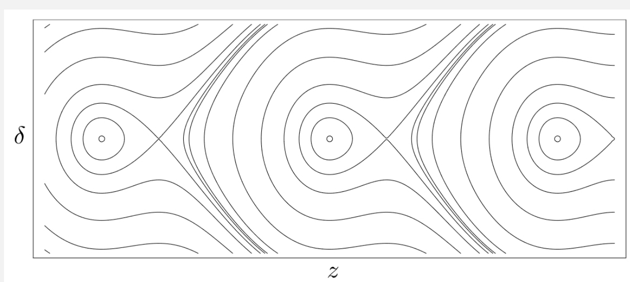


Figure 1: Longitudinal phase space portrait for a synchrotron storage ring (Wolski)

Stable Fixed Point (SFP)
Unstable Fixed Point (UFP)

ν_s for arbitrary amplitude

For a specific particle, we have

- $H_0 = H(z, \delta)$
- $\delta = 0$, get z_{min} and z_{max}
- $z = 0$, get δ_{max}

With

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{1}{\beta_0^2}\eta_p \delta \quad \Rightarrow \quad ds = -\frac{dz}{\eta_p \delta}\beta_0^2 \tag{54}$$

The oscillation period (in path length) for a particle is the double of the path length which travel from z_{min} to z_{max} is

$$L(H_0) = \int ds = 2\int_{z_{\min}}^{z_{\max}} -\frac{\beta_0^2}{\eta_p \delta(H_0, z)} dz \tag{55}$$

where

$$\delta(H_0, z) = \sqrt{2\beta_0^2 \cdot \frac{1}{\eta_p}\left\{\frac{qV_{rf}}{\omega_{rf} P_0 C_0}\left[-\cos(\phi_s - \frac{\omega_{rf} z}{c})\right] + \sin\phi_s \cdot \frac{\omega_{rf} z}{c} - H_0\right\}} \tag{56}$$

Then the longitudinal tune

$$\nu_s(H_0) = \frac{C_0}{L(H_0)} \tag{57}$$

Assuming $\phi_s = 0$, (ref S.Y. Lee)

$$\nu_s(\hat{\phi}) \approx \nu_s \left(1 - \frac{\hat{\phi}^2}{16}\right) \tag{58}$$

where $\hat{\phi} = -\omega_{rf} z_{max}/c$.

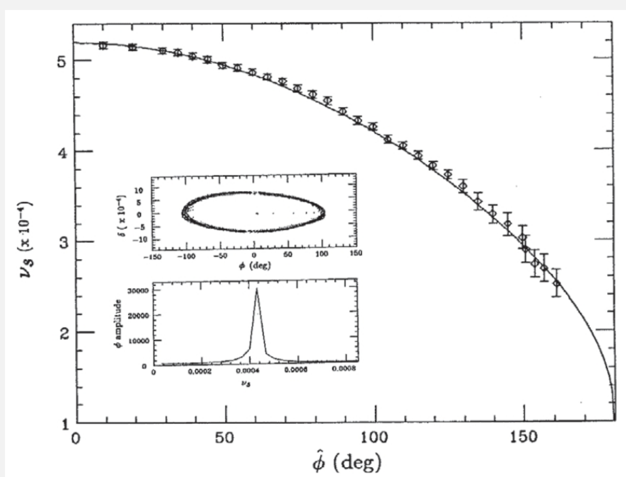


Figure 2: The measured synchrotron tune obtained by taking the FFT of the synchrotron phase coordinate is plotted as a function of the maximum phase amplitude of the synchrotron oscillations. The zero amplitude synchrotron tune was $\nu_s = 5.2 \times 10^{-4}$ (S.Y. Lee)

Small amplitude

$$
\begin{aligned}
H(z,\delta;s) &= \frac{qV_{rf}}{w_{rf}P_0C_0}\left[-\cos(\phi_s - \frac{w_{rf}z}{c}) + \sin\phi_s(\frac{w_{rf}z}{c})\right] - \frac{1}{2}\frac{1}{\beta_0^2}\eta_p\cdot\delta^2 \\
&\approx \frac{qV_{\mathrm{rf}}w_{\mathrm{rf}}\cos\phi_s}{2P_0C_0c^2}z^2 - \frac{1}{2}\frac{1}{\beta_0^2}\eta_p\delta^2
\end{aligned} \tag{59}
$$

where the static term has been droped.
We define the beta function

$$
\begin{aligned}
\beta_z = \frac{\hat{z}}{\hat{\delta}} &= \sqrt{\frac{P_0C_0c^2\cdot\eta_p}{qV_{\mathrm{rf}}w_{\mathrm{rf}}\cdot\cos\phi_s}} \\
&= \frac{\eta_p\cdot C_0}{\beta_0 2\pi\nu_s}
\end{aligned} \tag{60}
$$

Action/Angle

(ψ_z, J_z)?

$$
J_z = \frac{1}{2\pi}\oint\delta(H_0,z)dz = \frac{1}{\pi}\int_{z_{\min}}^{z_{\max}}\delta(H_0,z)dz \tag{61}
$$

H_0 is the number of Hamiltonian of the specific particle.
2nd type of generating function is used for the transformation from $(z,\delta)\to(\phi_z,J_z)$,

$$
F_2(z,J_z),\quad \delta = \frac{\partial F_2(z,J_z)}{\partial z},\quad \phi_z = \frac{\partial F_2(z,J_z)}{\partial J_z} \tag{62}
$$

Since we have

$$
\delta(H_0,z) = \pm\sqrt{-(H_0 - V(z))\cdot\frac{2\beta_0^2}{\eta_p}} \quad\Rightarrow \tag{63}
$$

$$
F_2(z,J_z) = \int_{z_{\min}}^{z} dz'\cdot\delta(H_0(J_z),z') \tag{64}
$$

Then we could get the one-to-one relationship between $(z,\delta)\leftrightarrow(\phi_z,J_z)$.
Don't forget

$$
\frac{d\phi_z}{ds} = \frac{\partial H(J_z)}{\partial J_z} \tag{65}
$$

Locality of RF Cavity

Recall that map through RF cavity

$$\Delta\delta = \frac{q}{P_0 c} V_{rf} \sin(\phi_s - \omega_{rf}\frac{z}{c}) - \frac{q}{P_0 c} V_{rf} \sin\phi_s \tag{66}$$

Map through arc

$$\Delta z = R_{56}\delta = -\frac{C_0}{\beta^2}\eta_p \delta \tag{67}$$

The linear map around the entrance of the cavity,

$$\begin{aligned} \begin{pmatrix} z \\ \delta \end{pmatrix}_{n+1} &= \begin{pmatrix} 1 & -\frac{C_0}{\beta_0^2}\eta_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{q}{P_0 c} V_{rf} \cos\phi_s \frac{\omega_{rf}}{c} & 1 \end{pmatrix} \begin{pmatrix} z \\ \delta \end{pmatrix}_n \\ &= \begin{pmatrix} 1 + \frac{C_0 \eta_p \omega_{rf}}{\beta_0^2 c} \frac{q V_{rf}}{P_0 c} \cos\phi_s & -\frac{C_0}{\beta_0^2}\eta_p \\ -\frac{q}{P_0 c}\frac{\omega_{rf}}{c} V_{rf} \cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} z \\ \delta \end{pmatrix}_n \end{aligned} \tag{68}$$

$$\frac{\mathrm{tr}M}{2} = 1 + \frac{\pi h \eta_p}{2\beta_0 P_0 c} \cdot q V_{rf} \cos\phi_s \tag{69}$$

The 'exact' tune considering locality of RF cavity is

$$v_{s,exact} = \frac{1}{2\pi}\cos^{-1}\left(\frac{\mathrm{Tr}M}{2}\right) = \frac{1}{2\pi}\cos^{-1}\left[1 - \frac{(2\pi\nu_s)^2}{2}\right] \tag{70}$$

The stability condition requires that

$$\nu_s < \frac{1}{\pi} \tag{71}$$

3 Selected topics

3.1 RF Phase Modulation (ref: S.Y. Lee)

Only a sinusoidal RF phase modulation is considered

$$\varphi = a \sin(\nu_m \theta + \chi_0) \tag{72}$$

With

- ν_m: modulation tune
- a: modulation amplitude
- χ_0: arbitrary phase factor
- θ: orbit angle

The Hamiltonian:

$$H(\phi, \mathcal{P}; \theta) = \underbrace{\frac{1}{2}\nu_s \mathcal{P}^2 + 2\nu_s \sin^2 \frac{\phi}{2}}_{H_0} + \underbrace{\nu_m a \mathcal{P} \cos(\nu_m \theta + x_0)}_{H_1} \tag{73}$$

ϕ is RF phase experienced by particle. We assume $\phi_s = 0$ and $\eta < 0$.

$$\mathcal{P} \equiv -\frac{h|\eta|}{\nu_s}\delta_p \quad \left(\delta_p \equiv \frac{\Delta P}{P_0}\right) \tag{74}$$

Reminder: $dt \cdot \omega_0 = d\theta$, $\nu_s = \sqrt{\frac{h|\eta|eV_{rf}}{2\pi\beta^2 E}}$)

Without Phase Modulation

Action

$$J(H_0) = \frac{1}{2\pi}\oint \mathcal{P}(\phi, H_0)d\phi \tag{75}$$

For small oscillation $J = \frac{1}{2}(\phi^2 + \mathcal{P}^2)$.
Tune

$$\nu_s(J) = \frac{\partial H_0}{\partial J} \approx \nu_s(1 - \frac{J}{8}\cdots) \tag{76}$$

For the canonical transformation

$$(\phi, \mathcal{P}) \to (\psi, J) \quad \texttt{Here we use } \psi \texttt{ instead of } \phi_z \tag{77}$$

The generating function

$$F_2(\phi, J) = \int_0^\phi \mathcal{P}(\phi', H_0(J))d\phi' \tag{78}$$

We could do Fourier expansion for $\mathcal{P}$ in (ψ, J)

$$\mathcal{P} = \sum_n f_n(J)e^{in\psi} \tag{79}$$

and get

$$\mathcal{P}(\psi, J) \approx (2J)^{1/2}\cos\psi + \frac{(2J)^{3/2}}{64}\cos 3\psi\cdots \tag{80}$$

Together with (the term in $H(\phi, \mathcal{P})$)

$$2\sin^2\left(\frac{\phi}{2}\right) = \sum_n G_n(J)e^{in\psi} \approx -\frac{J}{2}\cos 2\psi - \frac{J^2}{32}\cos 4\psi\cdots \tag{81}$$

We get

$$H_0(\psi, J; \theta) \approx \nu_s J - \frac{1}{16}\nu_s J^2 \tag{82}$$

With Modulation in (ψ, J)

The phase modulation term in Hamiltonian

$$\begin{aligned} H_1 &= \nu_m a\sqrt{J/2}\,[\cos(\psi + \nu_m\theta + \chi_0) + \cos(\psi - \nu_m\theta - \chi_0)] \\ &+ \nu_m a\frac{(2J)^{3/2}}{128}\,[\cos(3\psi + \nu_m\theta + \chi_0) + \cos(3\psi - \nu_m\theta - \chi_0)] + \ldots, \end{aligned} \tag{83}$$

Since $\mathcal{P}$ only contains odd frequency $(\psi, 3\psi, 5\psi, \ldots)$ (reminder: $H_1 = \nu_m a\mathcal{P}\cos(\nu_m\theta + x_0)$), H_1 only induce odd order parametric resonances:

$$\nu_m : \nu_s = 1:1 \quad \text{or} \quad 3:1 \quad \ldots \tag{84}$$

In the following, we only consider $\nu_m \approx \nu_s$.

$$H(\psi, J; \theta) \approx \nu_s J - \frac{1}{16}\nu_s J^2 + \frac{\nu_s a}{2}J^{1/2}\cos(\psi - \nu_m\theta - \chi_0) \tag{85}$$

In resonance rotating frame

to resonance rotating frame:$(\psi, J) \to (\chi, I)$ With generating function

$$F_2(\psi, I) = (\psi - \nu_m\theta - \chi_0 - \pi)I, \quad \Rightarrow \tag{86}$$

$$\chi = \psi - \nu_m\theta - \chi_0 - \pi, \quad I = J; \tag{87}$$

And the new Hamiltonian (time independent)

$$\tilde{H}(\chi, I; \theta) = (\nu_s - \nu_m)I - \frac{1}{16}\nu_s I^2 - \nu_s \frac{a}{\sqrt{2}} I^{1/2} \cos\chi. \tag{88}$$

A torus of particle motion will follow a constant Hamiltonian contour.
Fixed points of Hamiltonian:

$$\dot{I} = 0, \quad \dot{\chi} = 0. \quad (\chi = 0 \text{ or } \pi) \tag{89}$$

Define $g = \sqrt{2I} \cdot \cos\chi$ to represent the phase space coordinate of a fixed point.

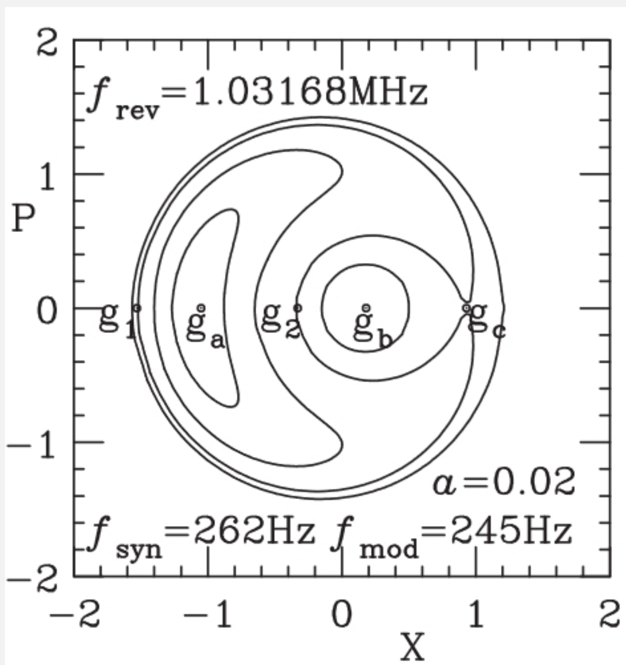


Figure 3: Poincare surfaces with RF phase modulation (SYLee). The SFPs are g_a and g_b, and the UFP is g_c. Here $X \equiv \sqrt{2I}\cos\chi$ and $P \equiv -\sqrt{2I}\sin\chi$

Measurment of RF phase modulation (S.Y. Lee)

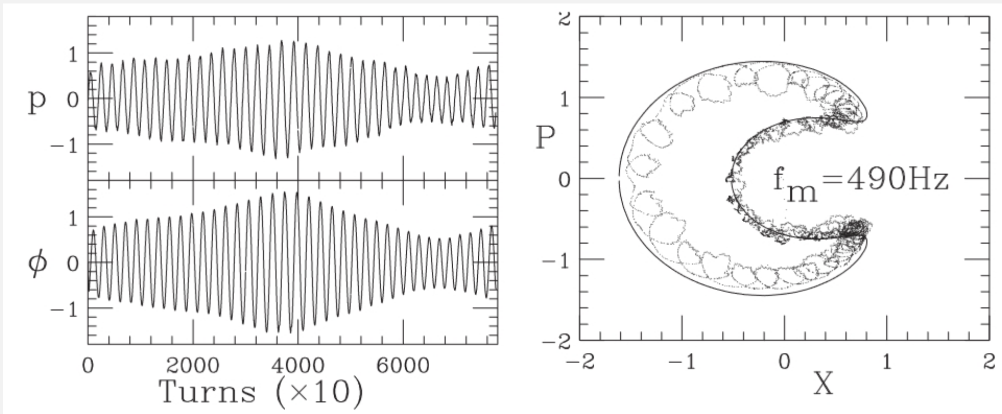


Figure 4: Left plots in normalized off-momentum coordinate $\mathcal{P}$ and the phase ϕ vs revolutions at 10-turn intervals. Right plots: the corresponding Poincare surfaces of section. The solid line shows the Hamiltonian torus. (SYLee) $P \equiv -\sqrt{2I}\sin\chi$

3.2 RF Voltage Modulation (ref: S.Y. Lee)

$\eta < 0$ and $\phi_s = 0$
Normalized phase space coordinates $(\phi, \mathcal{P})$

$$\mathcal{P} \equiv -\frac{h|\eta|}{\nu_s}\delta_p \quad \left(\delta_p \equiv \frac{\Delta P}{P_0}\right) \tag{90}$$

$$\nu_s = \sqrt{\frac{h|\eta| eV_{rf}}{2\pi\beta^2 E}} \tag{91}$$

The dynamics equation:

$$\begin{aligned} \frac{d\phi}{d\theta} &= \nu_s \mathcal{P} \\ \frac{d\mathcal{P}}{d\theta} &= \frac{\eta}{|\eta|}\nu_s \sin\phi \end{aligned} \tag{92}$$

$$\Rightarrow \quad \ddot{\phi} + \nu_s^2 \sin\phi = 0 \tag{93}$$

With RF voltage modulation:

$$\Delta V_{rf} = b \cdot V_{rf} \sin(\nu_m \theta + \chi) \tag{94}$$

$$\Rightarrow \quad \ddot{\phi} + \nu_s^2 \left[1 + b\sin(\nu_m\theta + \chi)\right]\sin\phi = 0 \quad (3.114) \tag{95}$$

Linear approximation with $\sin\phi \approx \phi$ (Gennady)

In linear approximation with $\sin\phi = \phi$, reduces to Mathieu equation (frequency is time-dependent),

$$\ddot{\phi} + \nu_s^2 \left[1 + b\sin(\nu_m\theta + \chi)\right]\phi = 0 \tag{96}$$

For small b, oscillations become unstable if $\frac{\nu_s}{\nu_m}$ is close to $\frac{n}{2}$ (n is integer),

$$\nu_m = 2\nu_s, \quad \nu_s, \quad \frac{2}{3}\nu_s, \quad \frac{1}{2}\nu_s, \quad \ldots \tag{97}$$

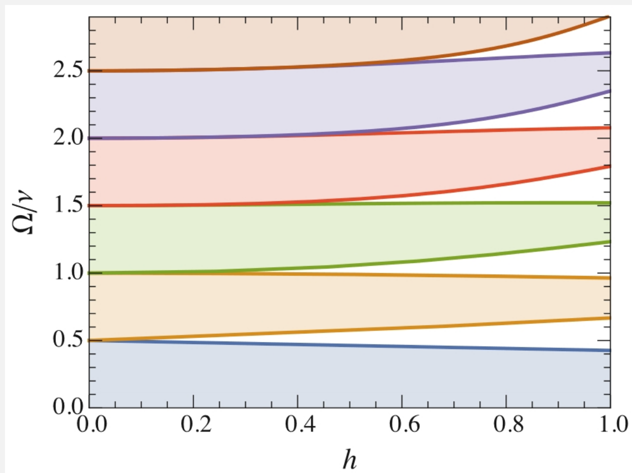


Figure 5: Stability regions for the Mathieu equation as functions of amplitude modulation $h = b$. The stable regions are shadowed. $\Omega/\nu \equiv \nu_m/\nu_s$. (Gennady)

Slow variation of $\omega_0(t)$

$$\ddot{x} + \omega_0^2(t)x = 0 \tag{98}$$

In time ω_0^{-1}, the relative change of ω_0 is small,

$$\omega_0^{-2}\left|\frac{d\omega_0}{dt}\right| \ll 1, \tag{99}$$

a solution as a real part of the complex function $\xi(t)$ is given by

$$\xi(t) = A(t)\exp\left(-i\int_0^t \omega_0(t')dt' + i\phi_0\right). \tag{100}$$

$A(t)$ is the slowly varying amplitude of the oscillation, ϕ_0 is the initial phase. Substituting this into equation,

$$\ddot{A} - 2i\omega_0\dot{A} - i\dot{\omega}_0 A = 0 \tag{101}$$

$\ddot{A}$ is neglected since A is a slow function of time,

$$\begin{aligned} &\Rightarrow \quad 2\omega_0\dot{A} + \dot{\omega}_0 A = 0 \\ &\Rightarrow \quad \frac{d}{dt}\ln(A^2\omega_0) = 0 \\ &\Rightarrow \quad A^2\omega_0 = \text{const} \end{aligned} \tag{102}$$

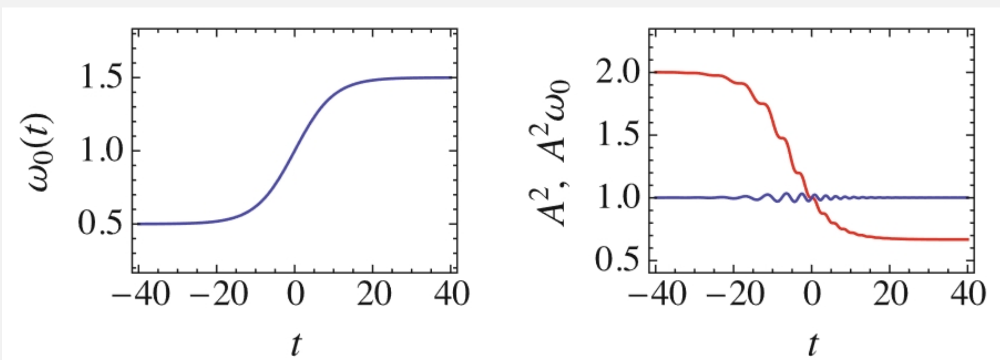


Figure 6: Adiabatic invariance of $A^2\omega_0$: the left panel shows the function $\omega_0(t)$; on the right panel the red curve shows the quantity $x(t)^2 + \dot{x}^2(t)/\omega_0^2(t)$ (which is close to the amplitude squared, A^2) while the blue curve shows the product of this quantity with $\omega_0(t)$ (Gennady).

Parametric resonance due to Voltage Modulation

Perturbed Hamiltonian $H = H_0 + H_1$, with

$$\begin{aligned} H_0 &= \frac{1}{2}\nu_s \mathcal{P}^2 + \nu_s(1-\cos\phi) \\ H_1 &= \nu_s b \sin(\nu_m\theta + \chi)[1-\cos\phi] \end{aligned} \tag{103}$$

Fourier expansion of H_1 in action-angle variables (ψ, J),

$$H_1(\psi, J; \theta) = \nu_s b \sum_{n=-\infty}^{\infty} |G_n(J)| \sin(\nu_m\theta - n\psi - \gamma_n) \tag{104}$$

where $\chi = 0$ for simplicity. γ_n is phase of $G_n(J)$.
Recall that

$$2\sin^2\left(\frac{\phi}{2}\right) = \sum_n G_n(J) e^{in\psi} \tag{105}$$

Only even n exist,

$$\begin{aligned} G_0 &\approx \frac{1}{2}J + \frac{1}{2048}J^3 + \cdots \quad \Rightarrow \Delta\nu_s = \frac{\partial H_1}{\partial J} \approx \frac{1}{2}\nu_s b \sin\nu_m\theta \\ G_2 &\approx -\frac{1}{4}J + \frac{1}{128}J^2 + \cdots \\ G_4 &\approx -\frac{1}{64}J^2 + \frac{1}{2048}J^3 + \cdots \end{aligned} \tag{106}$$

If $\nu_m = n\nu_s$, particle motion can be coherently perturbed by the RF voltage modulation resulting from a resonance driving term. The system is most sensitive to the RF voltage modulation at the second synchrotron harmonic.

ν_m=$n\nu_s$

From (ψ, J) (action-angle coordinates of the unperturbed Hamiltonian)
to $(\tilde{\psi}, \tilde{J})$ (resonance rotating frame)
using generating function

$$F_2 = \left(\psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n} + \frac{\pi}{2n}\right)\tilde{J} \quad \Rightarrow \tag{107}$$

$$\tilde{\psi} = \psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n} + \frac{\pi}{2n}; \quad \tilde{J} = J. \tag{108}$$

And Hamiltonian $\tilde{H}(\tilde{\psi}, \tilde{J}; \theta) = H_0 + H_1 + \frac{\partial F_2}{\partial \theta}$,

$$\begin{aligned} H_0(\tilde{\phi}, \tilde{J}) &= \frac{1}{2}\nu_s \mathcal{P}^2 + \nu_s(1 - \cos\phi) \\ &\approx \nu_s J - \frac{1}{16}\nu_s J^2 \\ &= \nu_s \tilde{J} - \frac{1}{16}\nu_s \tilde{J}^2 \end{aligned} \tag{109}$$

$$\begin{aligned} H_1(\tilde{\phi}, \tilde{J}; \theta) &= \nu_s b \cdot |G_n(J)| \cdot \sin(\nu_m\theta - n\psi - \gamma_n) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m\theta) \\ &= \nu_s b |G_n(J)| \sin\left[-n\tilde{\psi} + \frac{\pi}{2}\right] + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m\theta) \\ &= \nu_s b |G_n(\tilde{J})| \cdot \cos(n\tilde{\psi}) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m\theta) \end{aligned} \tag{110}$$

$$\frac{\partial F_2}{\partial \theta} = -\frac{\nu_m}{n}\tilde{J} \tag{111}$$

Then we get

$$\begin{aligned} \tilde{H}(\tilde{\psi}, \tilde{J}; \theta) &= H_0 + H_1 + \frac{\partial F_2}{\partial \theta} \\ &= \nu_s \tilde{J} - \frac{1}{16}\nu_s \tilde{J}^2 - \frac{\nu_m}{n}\tilde{J} \\ &\quad + \nu_s b |G_n(\tilde{J})| \cdot \cos(n\tilde{\psi}) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m\theta) \end{aligned} \tag{112}$$

$\nu_m \approx 2\nu_s$

with time average and $n = 2$,

$$\langle \tilde{H}(\tilde{\psi}, \tilde{J}; \theta) \rangle \approx (\nu_s - \frac{\nu_m}{2})\tilde{J} - \frac{\nu_s}{16}\tilde{J}^2 - \frac{\nu_s}{4} b\tilde{J}\cos(2\tilde{\psi}) \tag{113}$$

Now $\langle H \rangle$ is time independent.
Fixed Points:

$$\begin{aligned} \dot{\tilde{J}} &= \frac{\nu_s}{2} b\tilde{J}\sin 2\tilde{\psi} = 0 \\ \dot{\tilde{\psi}} &= \nu_s - \frac{\nu_m}{2} - \frac{\nu_s}{8}\tilde{J} + \frac{\nu_s}{4} b\cos 2\tilde{\psi} = 0 \end{aligned} \tag{114}$$

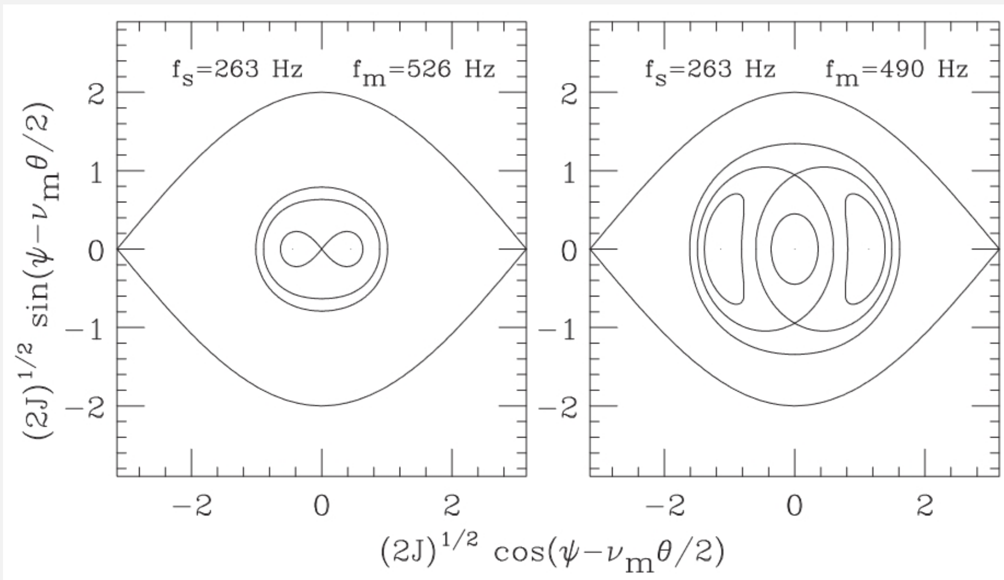


Figure 7: Separatrix and tori of the Hamiltonian in the resonance rotating frame. The voltage modulation amplitude is b =0.05. (S.Y. Lee)

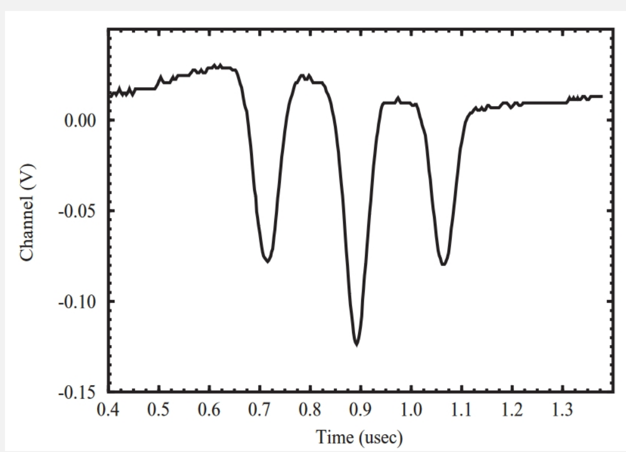


Figure 8: The beam bunch was observed to split into three beamlets in a single rf bucket. The voltage modulation amplitude is b =0.05 at modulation frequency f_m = 480 Hz with synchrotron tune f_s = 263 Hz. Note that the outer two beamlets rotated around the center beamlet at a frequency equal to half the modulation frequency. (S.Y. Lee)

3.3 Harmonic Cavity

Higher Harmonic cavity ⇒
Flatten the potential well ⇒
Bunch Lengthening

- weaken space change
- increase IBS time in light source
- mitigate X-Z instability in e+e- collider

For single RF system (orbit angle θ as time),

$$\dot{\delta} = \frac{eV_{rf}}{2\pi\beta^2 E}\left[\sin\phi - \sin\phi_s\right] \tag{115}$$

- ϕ is the phase coordinate relative to the primary RF cavity
- ϕ_s is the phase angle for the synchronous particle
- V_{rf} is the primary rf cavity voltage

For double RF system

$$\dot{\delta} = \frac{e}{2\pi\beta^2 E}\left\{V_1\left[\sin\phi - \sin\phi_{1s}\right] + V_2\left[\sin(\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})) - \sin\phi_{2s}\right]\right\} \tag{116}$$

$$\dot{\phi} = h_1 \eta \delta \tag{117}$$

The Hamiltonian

$$\begin{aligned} H(\phi, \delta; \theta) = \frac{1}{2}h_1\eta\delta^2 + \frac{e}{2\pi\beta^2 E}\Big\{ & V_1\left[\cos\phi - \cos\phi_{1s} + (\phi - \phi_{1s})\cdot\sin\phi_{1s}\right] \\ & + \frac{h_1}{h_2}V_2\left[\cos(\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})) - \cos\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})\cdot\sin\phi_{2s}\right]\Big\} \end{aligned} \tag{118}$$

With $h = \frac{h_2}{h_1}$, $r = \frac{V_2}{V_1}$ and in normalized momentum $\mathcal{P} = -\frac{h_1|\eta|}{\nu_s}\delta$ $\quad (\nu_s = \sqrt{\frac{h_1 e V_1 |\eta|}{2\pi\beta^2 E_0}}$, assuming $\eta < 0$),

$$H(\phi, \mathcal{P}; \theta) = \frac{1}{2}\nu_s \mathcal{P}^2 + \nu_s \cdot \Big\{ (\cos\phi_{1s} - \cos\phi) + (\phi_{1s} - \phi)\sin\phi_{1s} + \frac{r}{h} \cdot [\cos\phi_{2s} - \cos(\phi_{2s} + h(\phi - \phi_{1s})) - h(\phi - \phi_{1s})\sin\phi_{2s}] \Big\} \tag{119}$$

A flattened potential well requires

$$\frac{\partial H}{\partial \phi} = \nu_s \left\{ \sin\phi - \sin\phi_{1s} + \frac{r}{h} \cdot [h \sin(\phi_{2s} + h(\phi - \phi_{1s})) - h\sin\phi_{2s}] \right\} = 0 \tag{120}$$

$$\Rightarrow \quad \phi = \phi_{1s} \tag{121}$$

$$\frac{\partial^2 H}{\partial \phi^2} = \nu_s \{ \cos\phi + rh\cos(\phi_{2s} + h(\phi - \phi_{1s})) \} = 0 \tag{122}$$

$$\Rightarrow \quad \cos\phi_{1s} + rh\cos\phi_{2s} = 0 \tag{123}$$

$$\frac{\partial^3 H}{\partial \phi^3} = \nu_s \cdot \left\{ -\sin\phi - rh^2 \sin(\phi_{2s} + h(\phi - \phi_{1s})) \right\} = 0 \tag{124}$$

$$\Rightarrow \quad \sin\phi_{1s} + r \cdot h^2 \cdot \sin\phi_{2s} = 0 \tag{125}$$

Example of harmonic cavity configuration

$h = 2$ and $\phi_{1s} = \phi_{2s} = 0°$

$$H(\phi, \mathcal{P}; \theta) = \frac{\nu_s}{2}\mathcal{P}^2 + \nu_s \left\{ (1 - \cos\phi) + \frac{r}{2}[1 - \cos 2\phi] \right\} \tag{126}$$

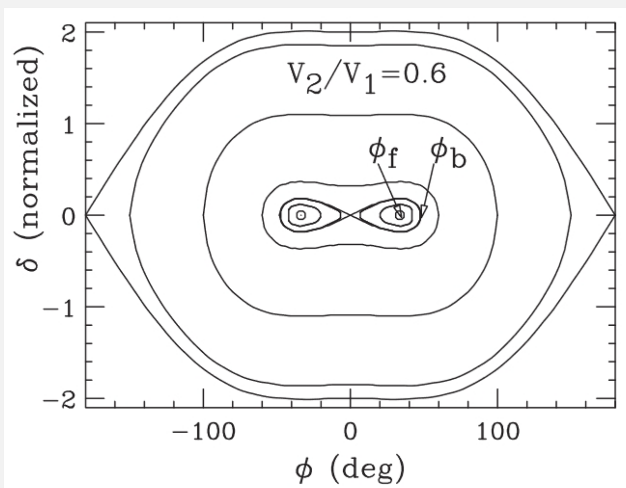


Figure 9: RF bucket and phase space ellipses for a double rf system with $h_2/h_1 = 2$ and $r = V_2/V_1 = 0.6$. (S.Y. Lee)

3.4 Nonlinear Phase Slip Factor

With

$$\eta = \eta_0 + \eta_1\delta + \ldots \tag{127}$$

$$\frac{d\phi}{d\theta} = -\frac{\partial H}{\partial \delta} = h \cdot \eta(\delta) \cdot \delta = h\eta_0\delta + h\eta_1\delta^2 \tag{128}$$

Hamiltonian

$$\begin{aligned} H(\phi, \delta; \theta) = \frac{1}{2}h\left(\eta_0 + \frac{2}{3}\eta_1\delta\right)\delta^2 \\ + \frac{eV}{2\pi\beta^2 E}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] \end{aligned} \tag{129}$$

Fixed Points:

$$\begin{aligned} (\phi, \delta)_{SFP} = (\phi_s, 0), \quad (\pi - \phi_s, -\eta_0/\eta_1) \\ (\phi, \delta)_{UFP} = (\pi - \phi_s, 0), \quad (\phi_s, -\eta_0/\eta_1) \end{aligned} \tag{130}$$

With $\nu_s = \sqrt{\frac{heV|\eta_0|}{2\pi\beta^2 E_0}}$,
in normalized phase space corrdinates ϕ and $\mathcal{P} = \frac{h|\eta_0|}{\nu_s}\delta$ (assuming $\eta_0 > 0$, $\eta_1 > 0$),
Hamiltonian

$$H(\phi, \mathcal{P}; \theta) = \frac{1}{2}\nu_s\mathcal{P}^2 + \frac{1}{2y}\nu_s P^3 + \nu_s\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] \tag{131}$$

where $y \equiv \frac{3h\eta_o^2}{2\eta_1\nu_s}$.
If $|y| \gg 1$, nonlinear phase slip factor is not important, otherwise phase space tori will be deformed.
Two separatrix will cross if

$$H(\phi = \pi - \phi_s, \mathcal{P} = 0) = H(\phi = \phi_s, \mathcal{P} = -\frac{h|\eta_0|}{\nu_s}\frac{\eta_0}{\eta_1}) \tag{132}$$

where $y = y_{cr}$,

$$y_{cr} = \sqrt{27\left[\left(\frac{\pi}{2} - \phi_s\right)\sin\phi_s - \cos\phi_s\right]} \tag{133}$$

Example of non-linear phase slip factor.

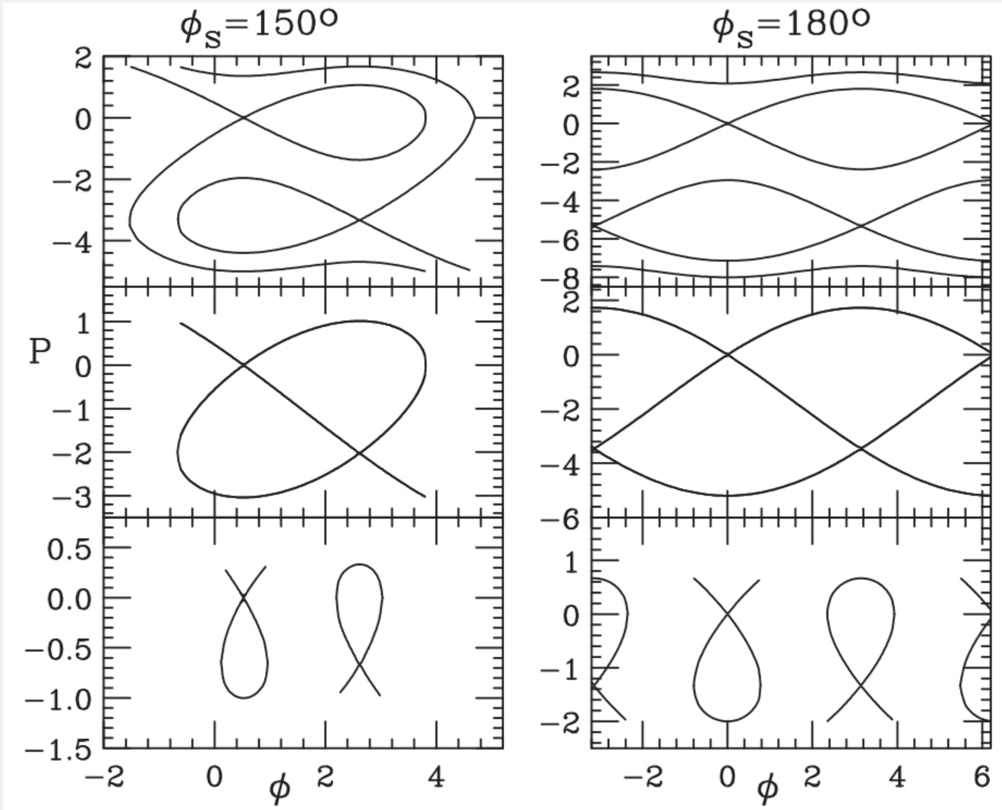


Figure 10: Separatrix in the normalized phase space. Left: $\phi_s = 150°$. $y = 5$ (top), $y = y_{cr} = 3.0406$ (middle) and 1 (bottom). Right: $\phi_s = 150°$. $y = 8$ (top), $y = y_{cr} = 5.1962$ (middle), and 3 (bottom). (S.Y. Lee)

3.5 Local Phase Slippage (ref: Deng, PRAB 24, 094001 (2021)

One turn map in Courant-Snyder Formalism

$$M_z = \begin{pmatrix} \cos\Phi_z + \alpha_z \sin\Phi_z & +\beta_z \sin\Phi_z \\ -\gamma_z \sin\Phi_z & \cos\Phi_z + \alpha_z \sin\Phi_z \end{pmatrix} \tag{134}$$

where $\Phi_z = 2\pi\nu_s$.
The second moments

$$\Sigma_z = \begin{pmatrix} \langle z^2\rangle & \langle z\delta\rangle \\ \langle z\delta\rangle & \langle\delta^2\rangle \end{pmatrix} = \begin{pmatrix} \epsilon_z\beta_z + \epsilon_x\mathcal{H}_x & -\epsilon_z\alpha_z \\ -\epsilon_z\alpha_z & \epsilon_z\gamma_z \end{pmatrix} \tag{135}$$

ϵ_z is the longitudinal emittance.

$$\mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D'^2_x \tag{136}$$

At the middle of RF cavity, the one-turn map (assume no dispersion at RF)

$$M = \begin{pmatrix} 1 & 0 \\ \frac{k}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\eta C_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{k}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{k}{2}\eta C_0 & -\eta C_0 \\ k - (\frac{k}{2})^2\eta C_0 & 1 - \frac{k}{2}\eta C_0 \end{pmatrix} \tag{137}$$

where $k = \frac{eV_{rf}}{\beta^2 E_0} \cdot \cos\phi_s \cdot \frac{2\pi h}{C_0}$ (h : harmonic number, C_0: circumference)
In the conventional case,

$$\Phi_z \approx \begin{cases} -\sqrt{k\eta C_0} & \text{, if } \eta > 0 \\ \sqrt{k\eta C_0} & \text{, if } \eta < 0 \end{cases} \tag{138}$$

$$\beta_z = \frac{M_{12}}{\sin\Phi_z} \approx \sqrt{\frac{\eta C_0}{k}} = \frac{\eta C_0}{\Phi_z} \tag{139}$$

if $|\eta|$ is small, partial phase slippage may be larger than global.
variation of β_z
partial R_{56} from s_1 to s_2,

$$F(s_1, s_2) = -\int_{s_1}^{s_2} \left(\frac{D_x(s)}{\rho(s)} - \frac{1}{\gamma^2} \right) ds \tag{140}$$

at position s_j,

$$F(s_j, s_{rf-}) + F(s_{rf+}, sj) = -\eta C_0. \tag{141}$$

The transfer matrix

$$\begin{aligned} T(s_j, s_{rf-}) &= \begin{pmatrix} 1 & F(s_j, s_{rf-}) \\ 0 & 1 \end{pmatrix} \\ T(s_{rf-}, s_{rf+}) &= \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \\ T(s_{rf+}, s_j) &= \begin{pmatrix} 1 & F(s_{rf+}, s_j) \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{142}$$

One turn map at s_j,

$$M(s_j) = T(s_{rf+}, s_j) T(s_{rf-}, s_{rf+}) T(s_j, s_{rf-}) \tag{143}$$

$$\beta_z(s_j) = \frac{M_{12}(s_j)}{\sin \Phi_z} == \frac{-\eta C_0 + F(s_{rf+}, s_j) \cdot F(s_j, s_{rf-}) \cdot k}{\sin \Phi_z} \tag{144}$$

More exact longitudinal emittance:

$$\epsilon_z = \frac{5}{96\sqrt{3}} \frac{\alpha_F \cdot \lambda_e^2 \cdot \gamma^5}{\alpha_L} \oint \frac{\beta_z(s)}{|\rho(s)|^3} ds \tag{145}$$

where $\alpha_F = 1/137$ the fine structure constant, $\lambda_e = \lambda/2\pi = 386$ fm the reduced Compton wavelength of electron, $\alpha_L = J_z U_0/2E_0$ the longitudinal damping constant (J_z Longitudinal damping partition).

Example of local β_z and partial phase slip.

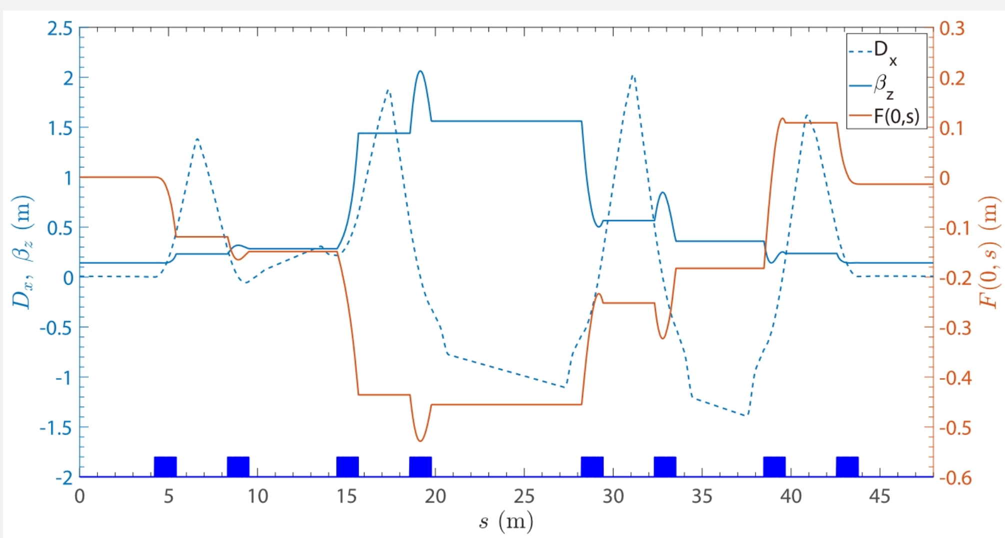


Figure 11: The horizontal dispersion D_x, longitudinal beta function β_z, and $F(0, s)$ of the MLS lattice. In this plot, the zero-length rf is placed at $s_{rf} = 0$ m and $V_{rf} = 80$ MV is applied. (X.J. Deng)

3.6 Synchrotron Radiation

With SR energy loss

$$U_\delta = U_0(1+\delta)^3 \tag{146}$$

The dynamics equation becomes

$$\begin{aligned} \frac{dz}{dt} &= -c\eta\delta \\ \frac{d\delta}{dt} &= \frac{1}{E_0 T_0}[eV(z) - U_0(1+\delta)^3] \end{aligned} \tag{147}$$

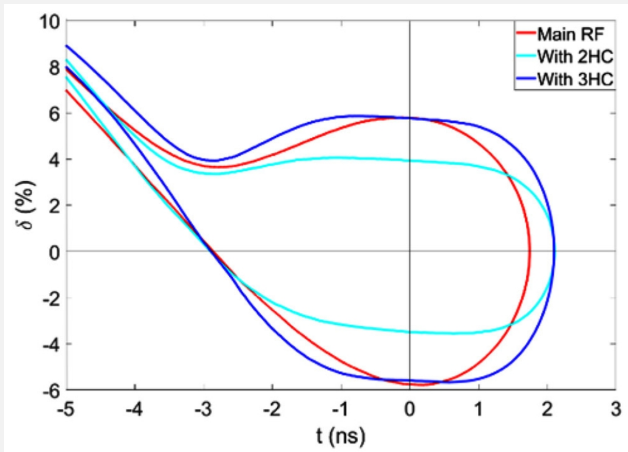


Figure 12: Longitudinal acceptance obtained by ELEGANT (NST (2019) 30:113)

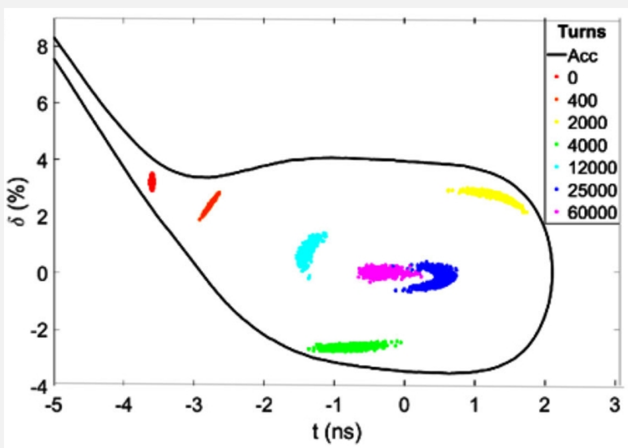


Figure 13: Longitudinal distribution during injection (NST (2019) 30:113)

3.7 Adidabatic Damping

Now E_s, β_s, ω_0, ϕ_s, V_0 and η depends on time t ,

$$\begin{aligned}(\text{Ampl. of } \tau) &\propto \left(\frac{-\eta}{\beta_s^2 E_s \omega_0^2 e V_0 \cos\phi_s}\right)^{\frac{1}{4}}, \\ (\text{Ampl. of } E - E_s) &\propto \left(\frac{-\eta}{\beta_s^2 E_s \omega_0^2 e V_0 \cos\phi_s}\right)^{-\frac{1}{4}}.\end{aligned} \tag{148}$$

The emittance in $(\tau, \Delta E)$ will keep unchanged.
reminder: $\Delta\phi = \omega_{rf}\tau$
In phase space coordinates

$$\begin{aligned} z &\equiv s - \tau\beta_s c \\ \delta_p &\equiv \frac{1}{\beta_s^2}\frac{\Delta E}{E_s}\end{aligned} \tag{149}$$

we have

$$(\text{Ampl . of } z) \times (\text{Ampl . of } \delta_p) \quad \propto \quad \frac{1}{\beta_s E_s} \times (\text{Ampl. of } \tau) \times (\text{Ampl. of } \Delta E) \tag{150}$$

In phase space coordinates

$$\begin{aligned} z &\equiv \frac{s}{\beta_s} - c\tau \\ \delta &\equiv \frac{\Delta E}{P_s c}\end{aligned} \tag{151}$$

we have

$$(\text{Ampl . of } z) \times (\text{Ampl . of } \delta) \quad \propto \quad \frac{1}{\beta_s E_s} \times (\text{Ampl. of } \tau) \times (\text{Ampl. of } \Delta E) \tag{152}$$

3.8 RF cavity at dispersion location

One turn arc map (only x and z) around RF cavity

$$M_{arc} = H^{-1}\begin{pmatrix} \cos\mu_x + \alpha\sin\mu_x & \beta_x \sin\mu_x & 0 & 0 \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x & 0 & 0 \\ 0 & 0 & 1 & -\eta C \\ 0 & 0 & 0 & 1 \end{pmatrix} H \tag{153}$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 & -D \\ 0 & 1 & 0 & -D' \\ D' & -D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{154}$$

The map through the RF cavity

$$M_{cav} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{pmatrix} \tag{155}$$

where $k = -\frac{qV_{rf} r w_{rf}}{P_0 c^2}\cos\phi_s$
The one turn map is $M = M_{arc}M_{cav}$.

The eigenvalues of M are $\lambda_1, 1/\lambda_1, \lambda_2, 1/\lambda_2$, we define

$$s = \lambda_1 + 1/\lambda_1 \text{ or } \lambda_2 + 1/\lambda_2 \tag{156}$$

s is the root of the following equation

$$s^2 - [2\cos\mu_x + 2\cos\mu_s + \mathcal{H}_x k \sin\mu_x] \cdot s + [4\cos\mu_x\cos\mu_s + 2\mathcal{H}_x k \sin\mu_x] = 0 \tag{157}$$

Don't forget

$$2\cos\mu_s = 2 - k\eta C \tag{158}$$

Reminder: $\mathcal{H}_x = \gamma_x D^2 + 2\alpha_x DD' + \beta D'^2$ which ever appear in the diffusion term in the evaluation of horizontal emittance due to SR.
It could be obtained that

$$\begin{aligned} s = &\left(\cos\mu_x + \cos\mu_s + \frac{1}{2}\mathcal{H}_x k \sin\mu_x\right) \\ &\pm \sqrt{\left(\cos\mu_x + \cos\mu_s + \frac{1}{2}\mathcal{H}_x k \sin\mu_x\right)^2 - (4\cos\mu_x\cos\mu_s + 2\mathcal{H}_x k \sin\mu_x)} \end{aligned} \tag{159}$$

Consider the case, $\eta > 0$, we have $k > 0$. When μ_x above integer, s would be complex (unstable) if $\cos\mu_x \approx \cos\mu_s$.

3.9 Crab Cavity

The map of crab cavity (only consider x and z)

$$M_{\text{crab}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_{\text{crab}} & 0 \\ 0 & 0 & 1 & 0 \\ k_{\text{crab}} & 0 & 0 & 1 \end{pmatrix} \tag{160}$$

where $k_{\text{crab}} = \frac{eV_{\perp,rf}\omega_{\perp,rf}}{cE_0}$. The crab RF voltage

$$V_\perp(t) = V_{\perp,rf}\sin(\omega_{\perp,rf}t) \approx V_{\perp,rf}\omega_{\perp,rf}\frac{z}{c} \tag{161}$$

The one turn map is

$$M_{\text{quadrf}} = M_{arc} \cdot M_{rf} \cdot M_{crab} \tag{162}$$

Here we assume no dispersion at cavity.
$e^{\pm\mu}$ is the eigenvalues of M

$$2\cos\mu = \cos\mu_s + \cos\mu_x \pm \sqrt{(\cos\mu_s - \cos\mu_x)^2 - \beta_x C_0 \eta k_{\text{crab}}^2 \sin\mu_x} \tag{163}$$

4 Spectrum

- Fourier transforms:
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{164}$$
In this equation $j = \sqrt{-1}$ and ω is the angular frequency
- The inverse transform is
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \tag{165}$$
- The image (wall) current i_W has a line density equal to the line density of the beam current i_b,
$$i_W(t) = -i_b(t) \text{ and } I_W(\omega) = -I_b(\omega) \tag{166}$$
with an overall minus sign since it is an image current.

A single particle with constant revolution frequency

- The current of a single, unit-charge particle with a constant revolution frequency ω_r, is a periodic set of impulses spaced a time $T = 2\pi/\omega_r$ apart

$$i_b(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{167}$$

- Fourier transforming gives

$$\begin{aligned} I_b(\omega) &= \int_{-\infty}^{\infty} i_b(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} e^{-j\omega nT} = \sum_{n=-\infty}^{\infty} e^{-j2\pi n\omega/\omega_r} \\ &= \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \end{aligned} \tag{168}$$

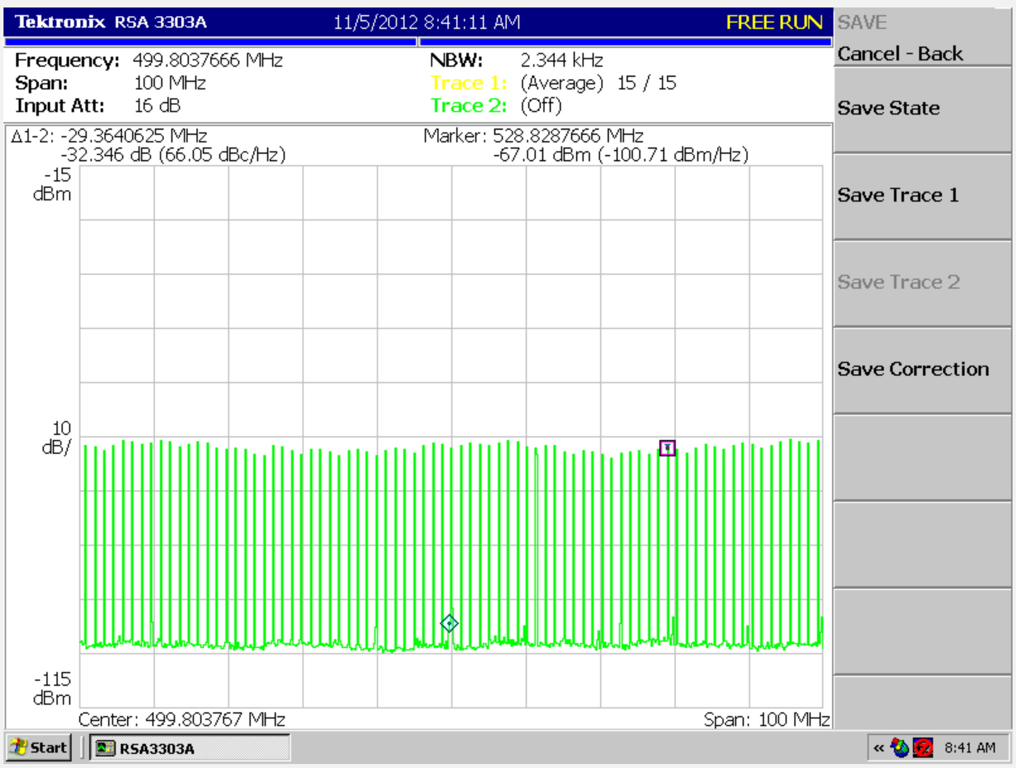


Figure 14: Single bunch spectrum obtained by spectrum analyzer (BEPCII).

Frequency Modulatiuon By Synchrotron Oscillations

- Synchrotron motion modulates the arrival time by

$$\tau = \tau_a \cos(\omega_s t + \psi) \tag{169}$$

where τ_a is the synchrotron oscillation amplitude, ω_s is the synchrotron frequency and ψ is the phase.

- The current is

$$i_b(t) = \sum_{n=-\infty}^{\infty} \delta(t - [nT + \tau_a \cos(\omega_s nT + \psi)])$$

- The Fourier transform is

$$I_b(\omega) = \sum_{n=-\infty}^{\infty} \exp(-j\omega[nT + \tau_a \cos(\omega_s nT + \psi)]) \tag{170}$$

- With a Bessel function expansion

$$e^{-jz\cos\theta} = \sum_{k=-\infty}^{\infty} J_k(z) e^{jk(\theta-\pi/2)} \tag{171}$$

The Fourier transform can be rewritten as

$$\begin{aligned} I_b(\omega) &= \sum_{n,k=-\infty}^{\infty} J_k(\omega\tau_a) e^{jk(\omega_s nT+\psi-\pi/2)} e^{-jn\omega T} \\ &= \sum_{n,k=-\infty}^{\infty} e^{jk(\psi-\pi/2)} J_k(\omega\tau_a) e^{-j2\pi n(\omega-k\omega_s)/\omega_r} \\ &= \omega_r \sum_{k=-\infty}^{\infty} e^{jk(\psi-\pi/2)} J_k(\omega\tau_a) \sum_{n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \end{aligned} \tag{172}$$

- For each rotation harmonic there is an infinite number of sidebands. They are displaced from the rotation harmonic by $k\omega_s$, $k = -\infty, \ldots, \infty$ and have different envelopes. The first maximum of J_k is at $\omega\tau_a \approx k$. When $|\omega| \ll 1/\tau_a$ only the rotation harmonics, $k = 0$, are present, and as the frequency increases more sidebands appear.

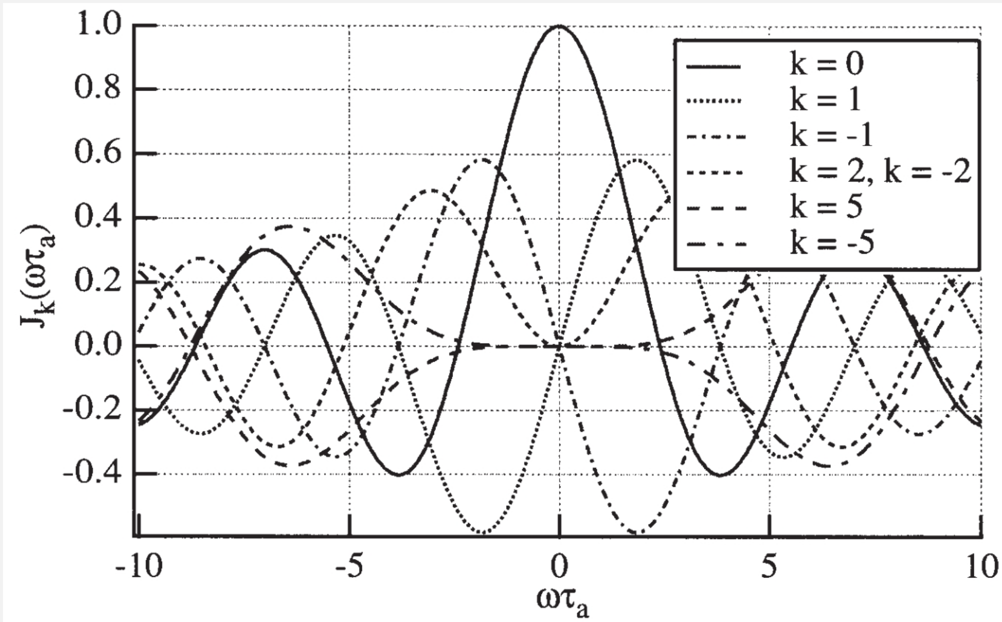


Figure 15: Envelope of synchrotron sideband.

Multiple Particles

With the longitudinal phase space density of the beam $\rho(\tau_a, \psi)$, where $\rho\tau_a d\tau_a d\psi$ is the charge in phase space area $\tau_a d\tau_a d\psi$, the longitudinal spectrum of a beam is

$$I(\omega) = \omega_r \sum_{k,n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \int_0^{2\pi} d\psi \rho(\tau_a, \psi) e^{jk\psi} J_k(\omega\tau_a) \tag{173}$$

If phase space density is $\rho(\tau_a, \psi) = \rho_0(\tau_a)/2\pi$, the beam signal is

$$\begin{aligned} I(\omega) &= \omega_r \sum_{k,n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \frac{1}{2\pi} \int_0^{2\pi} d\psi \rho_0(\tau_a) e^{jk\psi} J_k(\omega\tau_a) \\ &= \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \rho_0(\tau_a) J_0(\omega\tau_a) \end{aligned} \tag{174}$$

If the beam has charge Q and is Gaussian in τ with rms bunch length σ_τ,

$$\rho_0(\tau_a) = \frac{Q}{\sigma_\tau^2} \exp(-\tau_a^2/2\sigma_\tau^2) \tag{175}$$

and

$$I(\omega) = Q\omega_r \exp(-\omega^2\sigma_\tau^2/2) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \tag{176}$$

The appearance of synchrotron sidebands and azimuthal structure in longitudinal phase space are directly related.

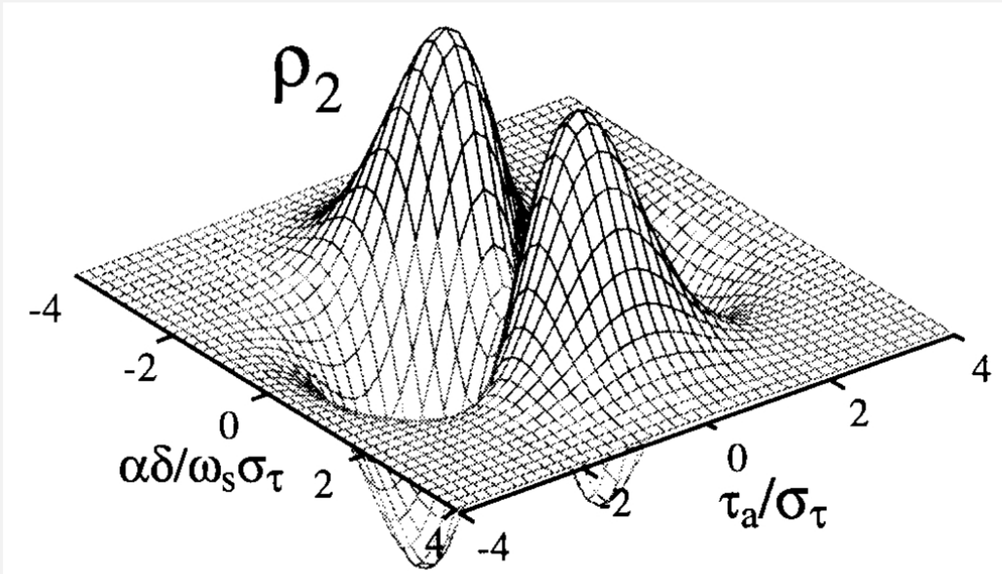
The phase space density can be written as a Fourier expansion

$$\rho(\tau_a, \psi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \rho_m(\tau_a) e^{im\psi} \tag{177}$$

Substituting into eq. (173),

$$\begin{aligned} I(\omega) &= \frac{\omega_r}{2\pi} \sum_{k,n,m=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \int_0^{2\pi} d\psi \rho_m(\tau_a) e^{j(k+m)\psi} J_k(\omega\tau_a) \\ &= \omega_r \sum_{m,n=-\infty}^{\infty} \delta(\omega + m\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \rho_m(\tau_a) J_k(\omega\tau_a) \end{aligned} \tag{178}$$

As a specific example, suppose that the beam has a quadrupole perturbation shown in the following figure.



$$\rho(\tau_a, \psi) = \frac{\rho_0(\tau_a)}{2\pi}\left(1 + A_2 \frac{\tau_a^2}{\sigma_\tau^2}\cos(2\psi)\right) \tag{179}$$

where ρ_0 is given by eq. (175). Substituting into (178) and performing the integrals

$$I(\omega) = Q\omega_r \exp(-\omega^2\sigma_\tau^2/2)\left[\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_r) + A_2\frac{\omega^2\sigma_\tau^2}{2}\left(\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_r + 2\omega_s) + \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_r - 2\omega_s)\right)\right] \tag{180}$$

The envelopes are shown in the following figure with $A_2 = 0.01$.