

Longitudinal Dynamics (OCPA 2025)

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A lot of references including below (but not all):

- Wolski - book
- A. Chao - book
- S.Y. Lee - book
- R. H. Siemann, Bunched beam diagnostics

- ...

Focus on circular machine
If any error, pls let me know.

1 Not so prerequisite

Configuration Space: 3D
Phase Space: 6D

Transverse: Dynamic Aperture (ring)
Longitudinal: Momentum Acceptance (ring)

Dynamic Coordinates

- x
- $p_x \approx \frac{dx}{ds}$
- $z = \frac{s}{\beta_0} - ct$. Different definition in different code. Here we follow Wolski (MADX?). In the note, maybe we don't keep same definition. (Sorry!)
- Instead of $p_z \approx \frac{dz}{ds}$, $p_z = \delta = \frac{\Delta E}{p_0 c}$. Since δ nearly constant along the ring.

Drift (Wolski)

$$H_{\text{drift}}(x, p_x, y, p_y, z, \delta; s) = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \quad (1)$$

$$\begin{cases} \frac{dp_x}{ds} = 0, \\ \frac{dx}{ds} = \frac{\partial H}{\partial p_x} = \frac{p_x}{D}, \\ \frac{d\delta}{ds} = 0, \\ \frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} \left(1 - \frac{1}{D}\right) - \frac{\delta}{D}, \end{cases} \Rightarrow \begin{cases} x_1 = x_0 + \frac{p_{x0}}{D} \cdot L, \\ y_1 = y_0 + \frac{p_{y0}}{D} \cdot L, \\ z_1 = z_0 + \frac{L}{\beta_0} \left(1 - \frac{1}{D}\right) - \frac{\delta}{D} \cdot L, \end{cases} \quad (2)$$

where $D \equiv \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}$

Linear Approximation of Drift Map

$$H \approx \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2 \gamma_0^2} \Rightarrow \begin{cases} x_1 = x_0 + L p_{x0}, \\ y_1 = y_0 + L p_{y0}, \\ z_1 = z_0 + \frac{L}{\beta_0^2 \gamma_0^2} \cdot \delta_0 \end{cases} \quad (3)$$

Quadrupole

With $K_1 = \frac{q}{p_0} \cdot \frac{\partial B_y}{\partial x}$,

$$H_Q = H_{\text{drift}} + \frac{K_1}{2}(x^2 - y^2), \quad (4)$$

$$H_Q \approx \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \frac{K_1}{2}(x^2 - y^2). \quad (5)$$

Linear Map:

$$M_{\text{Quad}} = \begin{pmatrix} \cos & \frac{\sin}{\sqrt{K_1}} & & & \\ -\sqrt{K_1} \sin & \cos & & & \\ & & \cosh & \frac{\sinh}{\sqrt{K_1}} & \\ & & \sqrt{K_1} \sinh & \cosh & \\ & & & & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ & & & & 0 & 1 \end{pmatrix}. \quad (6)$$

The parameter of sin, cos, sinh, cosh is $\sqrt{K_1}L$

It is well known that focus strength will be reduced for higher energy particle. This is not linear effect.

If only taking 2nd order approximation of p_x/p_y , but keep exact δ terms,

$$H_Q = \frac{p_x^2}{2D_\delta} + \frac{p_y^2}{2D_\delta} + \frac{\delta}{\beta_0} - D_\delta + \frac{K_1}{2}(x^2 - y^2), \quad (7)$$

where $D_\delta = \sqrt{1 + \frac{2\delta}{\beta_0} + \delta^2}$.

The transverse transfer matrix with chromatic effect,

$$\begin{pmatrix} \cos & \frac{\sin}{\sqrt{K'_1}D_\delta} & & & \\ -\sqrt{K'_1}D_\delta \sin & \cos & & & \\ & & \cosh & \frac{\sinh}{\sqrt{K'_1}D_\delta} & \\ & & \sqrt{K'_1}D_\delta \sinh & \cosh & \\ & & & & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ & & & & 0 & 1 \end{pmatrix} \quad (8)$$

where $K'_1 = \frac{K_1}{D_\delta}$ and the parameter of sin, cos, sinh, cosh is $\sqrt{K'_1}L$.

RF Cavity

A Pillbox Cavity, TM010

$$E_z = E_0 \cdot J_0(kr) \cdot \sin(\omega t + \phi_0) \quad (9)$$

$$B_\theta = \frac{E_0}{c} \cdot J_1(kr) \cdot \cos(\omega t + \phi_0) \quad (10)$$

where $\omega \approx 2.40483 \cdot \frac{c}{a}$ and $k = \frac{\omega}{c} = \frac{2.40483}{a}$. a is radius. Reminder: $J_0(2.40483) = 0$.

The vector potential is

$$\vec{A} = \left(0, 0, \frac{E_0}{\omega} \cdot J_0(kr) \cdot \cos(\omega t + \phi_0) \right) \quad (11)$$

With $z = \frac{s}{\beta_0} - ct$ and $\omega t = \frac{ks}{\beta_0} - kz$, Hamiltonian of Cavity is

$$H_{\text{cav}} = H_{\text{drift}} - \frac{q}{P_0} \cdot A_s, \quad r = \sqrt{x^2 + y^2} \quad (12)$$

H_{cav} depend on s !

For a simplified model, take average of H_{cav} over cavity length,

$$\begin{aligned} \langle H \rangle_{\text{cav}} &= \frac{1}{L} \int_{-L/2}^{+L/2} H_{\text{cav}} ds \\ &= H_{\text{drift}} - \frac{1}{L} \cdot \frac{q}{P_0} \cdot \frac{2E_0\beta_0}{c \cdot k^2} \cdot \sin\left(\frac{kL}{2\beta_0}\right) \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\ &= H_{\text{drift}} - \frac{qE_0}{P_0ck} \cdot T \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \end{aligned} \quad (13)$$

where transit time factor $T \equiv \frac{\sin\left(\frac{kL}{2\beta_0}\right)}{\frac{kL}{2\beta_0}}$

Kick approximation.

$L \rightarrow 0$, but $V_{\text{rf}} \equiv E_0 L T$ keep constant.

$$\begin{aligned}
 \langle H \rangle_{\text{cav}} L &= -\frac{q}{P_0 c k} \cdot E_0 \cdot L \cdot T \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\
 &= -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot J_0(kr) \cdot \cos(\phi_0 - kz) \\
 &\approx -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \cos(\phi_0 - kz)
 \end{aligned} \tag{14}$$

Approximation of small transverse amplitude is used in above equation.

2nd order approximation.

$$\langle H \rangle_{\text{cav}} L \approx -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \left[\cos(\phi_0) + \sin(\phi_0) \cdot kz - \frac{\cos(\phi_0) \cdot k^2 z^2}{2} \right] \tag{15}$$

The change of δ through the RF cavity is

$$\Delta\delta = -\frac{\partial(\langle H \rangle_{\text{cav}} L)}{\partial z} = \frac{q}{P_0 c} \cdot V_{\text{rf}} \cdot \sin(\phi_0) - \frac{q}{P_0 c} \cdot V_{\text{rf}} k \cos(\phi_0) \cdot kz \tag{16}$$

We could also get the map with finite L and truncate $\langle H \rangle$ with 2nd order approximation. See Wolski's book.

Change of reference momentum.

The reference momentum after RF cavity is not changed before. We could also change the reference momentum $P_0 \rightarrow P_1$. (CEPC). In this case, the conjugate momentum should be changed.

Recall

$$\begin{aligned} p_{x,0} &= \frac{\beta_x \gamma m c}{P_0}, \\ \delta_0 &= \frac{E}{c P_0} - \frac{1}{\beta_0} \end{aligned} \quad (17)$$

We have

$$\begin{aligned} p_{x0} \cdot P_0 &= p_{x1} \cdot P_1 \\ p_{y0} \cdot P_0 &= p_{y1} \cdot P_1 \\ \delta_1 &= \frac{P_0}{P_1} \delta_0 - \frac{1}{\beta_1} + \frac{\gamma_0}{\beta_1 \gamma_1} \end{aligned} \quad (18)$$

Matrix representation:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_1 = \begin{pmatrix} 1 & & & & & \\ & \frac{P_0}{P_1} & & & & \\ & & 1 & & & \\ & & & \frac{P_0}{P_1} & & \\ & & & & 1 & \\ & & & & & \frac{P_0}{P_1} \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_0 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\beta_1} + \frac{\gamma_0}{\beta_1 \gamma_1} \end{pmatrix} \quad (19)$$

For example, if ref momentum at s is determined, we could recalc the phase space coordinates with the local ref momentum. This could help "reduce" the number of (p_x, p_y, δ) . Small number is often preferred especially approximation is used.

The above transformation is non-symplectic.

How the emittance and Twiss function change with the reference momentum?

$$\Sigma = \langle \vec{x} \cdot \vec{x}^T \rangle = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \quad (20)$$

With

$$\vec{x}_1 = R \vec{x}_0 = \begin{pmatrix} 1 & 0 \\ 0 & P_0/P_1 \end{pmatrix} \vec{x}_0, \quad \Sigma_1 = R \cdot \Sigma_0 \cdot R^T \quad (21)$$

$$\begin{pmatrix} \beta_{x1} & -\alpha_{x1} \\ -\alpha_{x1} & \gamma_{x1} \end{pmatrix} \epsilon_{x1} = \begin{pmatrix} \beta_{x0} & -\frac{P_0}{P_1} \alpha_{x0} \\ -\frac{P_0}{P_1} \alpha_{x0} & \left(\frac{P_0}{P_1}\right)^2 \gamma_{x0} \end{pmatrix} \epsilon_{x0} \quad (22)$$

$$\epsilon_{x1} = \frac{P_0}{P_1} \cdot \epsilon_{x0} \implies \beta_1 \gamma_1 \epsilon_{x1} = \beta_0 \gamma_0 \epsilon_{x0} \quad (23)$$

$$\implies \begin{cases} \beta_{x1} = \frac{P_1}{P_0} \cdot \beta_{x0} \\ \alpha_{x1} = \alpha_{x0} \\ \gamma_{x1} = \frac{P_0}{P_1} \cdot \gamma_{x0} \end{cases} \quad (24)$$

example: CEPC Twiss function

Dipole

$h = \frac{1}{\rho}$ is the curvature of reference trajectory.

Normalized field strength $k_0 = \frac{q}{p_0} B_0$

Sacled vector potential: $\vec{a} = \frac{q}{p_0} \vec{A} = \left(0, 0, -k_0 x + \frac{k_0 h x^2}{2(1+hx)}\right)$

Approximated Hamiltonian (2nd order):

$$H_2 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2 \gamma_0^2} + \left(k_0 - h \left(1 + \frac{\delta}{\beta_0}\right)\right) x + \frac{k_0 h x^2}{2}. \quad (25)$$

With $h = k_0$ in most cases, the transfer matrix:

$$\begin{pmatrix} \cos\left(\frac{L}{\rho}\right) & \rho \sin\left(\frac{L}{\rho}\right) & \frac{\rho}{\beta_0} \left[1 - \cos\left(\frac{L}{\rho}\right)\right] \\ -\sin\left(\frac{L}{\rho}\right) / \rho & \cos\left(\frac{L}{\rho}\right) & \frac{1}{\beta_0} \sin\left(\frac{L}{\rho}\right) \\ & & 1 & L \\ & & 0 & 1 \\ -\frac{1}{\beta_0} \sin\left(\frac{L}{\rho}\right) & -\frac{\rho}{\beta_0} \left[1 - \cos\left(\frac{L}{\rho}\right)\right] & 0 & 0 & 1 & \frac{L}{\beta_0^2 \gamma_0^2} - \frac{1}{\beta_0^2} \left(L - \rho \sin\left(\frac{L}{\rho}\right)\right) \\ & & & & 0 & 1 \end{pmatrix} \quad (26)$$

For $h \neq k_0$ and with k_1 , see Wolski.

Path Length

Trajectory:

$$\vec{r}(s) = \vec{r}_0(s) + x\hat{x} + y\hat{y} \quad (27)$$

Length of trajectory:

$$\begin{aligned} \oint |d\vec{r}(s)| &= \oint \left| \hat{s}ds + x'\hat{x}ds + \frac{x}{\rho}ds\hat{s} + y'\hat{y}ds \right| \\ &= \oint ds \sqrt{(1 + \frac{x}{\rho})^2 + x'^2 + y'^2} \approx C + \oint ds \frac{x}{\rho} \end{aligned} \quad (28)$$

The path length difference of closed orbit

$$\Delta C = \oint \frac{x_{\text{COD}}(s)}{\rho(s)} ds$$

Considering the cod induced by dispersion:

$$\Delta C = \delta \oint \frac{D(s)}{\rho(s)} ds \quad (29)$$

$$\frac{\Delta C}{C} = \alpha_c \delta, \quad \text{or} \quad \alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds \quad (30)$$

Momentum compaction factor:

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds \quad (31)$$

Consider a storage ring which contains a horizontal kick with angle θ at s_0 . COD caused by this kick induces a change of orbital circumference:

$$\Delta C = \oint ds \frac{x_{COD}}{\rho(s)} = \theta \cdot D(s_0). \quad (32)$$

D is the dispersion.

With RF systems, ΔC has to be compensated by an adjustment of energy

$$\Delta\delta = -\frac{1}{\alpha_c} \frac{\Delta C}{C} = -\frac{1}{\alpha_c} \frac{\theta \cdot D(s_0)}{C} \quad (33)$$

A more accurate COD formula:

$$x_{COD}(s) = \theta \frac{\sqrt{\beta(s_0)\beta(s)}}{2 \sin \pi\nu} \cos(\pi\nu - |\psi(s) - \psi(s_0)|) - \theta \frac{D(s_0) \cdot D(s)}{\alpha_c \cdot C} \quad (34)$$

Symmetric DBA (ref S.Y. Lee)

$$\begin{aligned} \alpha_c &= \frac{\rho}{L_m} \left[\theta - \sin \theta + \frac{D_0 \sin \theta}{\rho} + D'_0(1 - \cos \theta) \right] \\ &= \frac{\rho\theta^3}{L_m} \left(\frac{1}{6} + \frac{D_0}{L\theta} + \frac{D'_0}{2\theta} \right) + \dots \end{aligned} \quad (35)$$

where

- D_0, D'_0 : entrance of dipole
- L_m : half length of DBA
- ρ : bending radius, $\theta = L/\rho$, L : length of dipole

With $D_0 = 0, D'_0 = 0$,

$$\alpha_c = \frac{\rho\theta^3}{6L_m} = \frac{\rho\theta^3}{6C\theta/(2\pi)} = \frac{\rho\theta^2}{6R} \quad (36)$$

α_c could be negative.

2 Basic Longitudinal dynamics

The potential well term of Hamiltonian contributed by Cavity,

$$H_{\text{cav}} = -\frac{q}{P_0 c k} \cdot V_{\text{rf}} \cdot \cos(\phi_0 - kz) \quad (37)$$

We ignore the length L in the following.

Sine we follow Wolski define

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} = \frac{E - E_0}{P_0 c} \quad (38)$$

And the momentum deviation

$$\delta_p = \frac{P - P_0}{P_0} \quad (39)$$

we have

$$\begin{aligned} \left. \frac{d\delta}{d\delta_p} \right|_{\delta_p=0} &= \beta_0 \\ \left. \frac{d}{d\delta_p} \right|_{\delta_p=0} &= \beta_0 \left. \frac{d}{d\delta} \right|_{\delta=0} \end{aligned} \quad (40)$$

$\Delta\delta$ through RF cavity

$$\Delta\delta = -\frac{\partial H}{\partial z} = \frac{q}{P_0 c} \cdot V_{\text{rf}} \sin(\phi_0 - \omega_{\text{rf}} \cdot \frac{z}{c}) \quad (41)$$

Considering energy loss U_0 due to SR, $\Delta\delta$ in one turn,

$$\Delta\delta = \frac{q}{P_0 c} \cdot V_{\text{rf}} \sin(\phi_0 - \omega_{\text{rf}} \cdot \frac{z}{c}) - \frac{U_0}{P_0 c} \quad (42)$$

Take the average in the whole ring

$$\frac{d\delta}{ds} = \frac{q V_{\text{rf}}}{c P_0 C_0} \cdot \sin(\phi_0 - \omega_{\text{rf}} \cdot \frac{z}{c}) - \frac{U_0}{c P_0 C_0} \quad (43)$$

With momentum compaction factor

$$\alpha_p = \frac{1}{C_0} \left. \frac{dC}{d\delta_p} \right|_{\delta_p=0} = \frac{1}{C_0} \int_0^{C_0} \frac{\eta_x}{\rho} ds \quad (44)$$

Phase slip factor

$$\begin{aligned} T &= \frac{C}{\beta c} \Rightarrow \\ \frac{1}{T} \frac{dT}{d\delta_p} &= \frac{1}{C} \frac{dC}{d\delta_p} - \frac{1}{\beta} \frac{d\beta}{d\delta_p} \Rightarrow \\ \eta_p &= \left. \frac{1}{T_0} \frac{dT}{d\delta_p} \right|_{\delta_p=0} = \alpha_p - \frac{1}{\gamma_0^2} \end{aligned} \quad (45)$$

α_p is the property of lattice.

Change design momentum of lattice, magnet strength is scaled and α_p will keep unchanged.

γ_0 the design beam energy

Transition energy:

$$\eta_p = 0 \quad \Rightarrow \quad \gamma_T = \frac{1}{\sqrt{\alpha_p}} \quad (46)$$

With $z = \frac{s}{\beta_0} - ct$,

at $s = s_0$ and $t = t_0$,

after one turn back at $s = s_0$, $t = t_0 + T$ (no dispersion at s_0)

$$\Delta z = \frac{C_0}{\beta_0} - cT \quad (47)$$

We have

$$R_{56} = \frac{\partial z}{\partial \delta} = -c \frac{dT}{d\delta} = -c \frac{1}{\beta_0} \frac{dT}{d\delta_p} = -\frac{C}{\beta_0^2} \eta_p \quad (48)$$

Average Δz over the ring,

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{T}{\beta_0 T_0} = -\frac{1}{\beta_0} \eta_p \delta_p = -\frac{1}{\beta_0^2} \eta_p \delta \quad (49)$$

Combine $\frac{d\delta}{ds}$ and $\frac{dz}{ds}$, we get

$$\begin{aligned}\frac{d^2z}{ds^2} &= -\frac{1}{\beta_0^2} \cdot \eta_p \cdot \frac{d\delta}{ds} \\ &= -\frac{1}{\beta_0^2} \cdot \eta_p \cdot \frac{qV_{rf}}{cP_0C_0} \sin(\phi_0 - \omega_{rf}\frac{z}{c}) + \frac{1}{\beta_0^2} \cdot \eta_p \cdot \frac{U_0}{cP_0C_0}\end{aligned}\quad (50)$$

If $\phi_0 = \phi_s$, $\sin \phi_s = \frac{U_0}{qV_{rf}}$, take the first order approximation of z in RHS,

$$\frac{d^2z}{ds^2} + \frac{1}{\beta_0^2} \frac{qV_{rf}}{cP_0C_0} \frac{\omega_{rf}}{c} \eta_p \cos \phi_s \cdot z = 0 \quad (51)$$

$\Rightarrow$ longitudinal tune

$$\begin{aligned}v_s &= \frac{C_0}{2\pi} \sqrt{-\frac{1}{\beta_0^2} \frac{qV_{rf}}{cP_0C_0} \frac{\omega_{rf}}{c} \eta_p \cos \phi_s} = \sqrt{-\frac{1}{2\pi} \frac{1}{\beta_0} \frac{qV_{rf}}{cP_0} \cdot h \cdot \eta_p \cos \phi_s} \\ &= \sqrt{-\frac{1}{2\pi} \cdot \frac{1}{\beta_0^2} \frac{qV_{rf}}{E_0} \cdot h \cdot \eta_p \cos \phi_s}\end{aligned}\quad (52)$$

where $\omega_{rf} = h\omega_0$ is used.

Hamiltonian of Longitudinal Oscillation

With equation of $\frac{dz}{ds}$ and $\frac{d\delta}{ds}$, the Hamiltonian could be obtained

$$H(z, \delta; s) = \frac{qV_{rf}}{w_{rf}P_0C_0} \left[-\cos\left(\phi_s - \frac{w_{rf}z}{c}\right) + \sin\phi_s\left(\frac{w_{rf}z}{c}\right) \right] - \frac{1}{2} \frac{1}{\beta_0^2} \eta_p \cdot \delta^2 \quad (53)$$

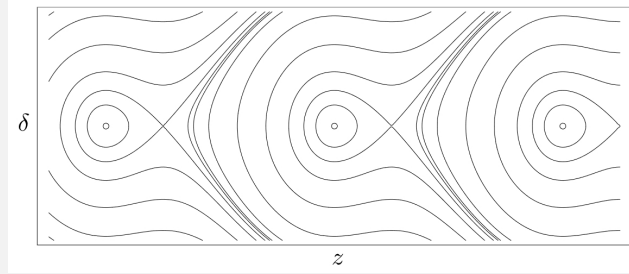


Figure 1: Longitudinal phase space portrait for a synchrotron storage ring (Wolski)

Stable Fixed Point (SFP)

Unstable Fixed Point (UFP)

ν_s for arbitrary amplitude

For a specific particle, we have

- $H_0 = H(z, \delta)$
- $\delta = 0$, get z_{min} and z_{max}
- $z = 0$, get δ_{max}

With

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{1}{\beta_0^2} \eta_p \delta \Rightarrow ds = -\frac{dz}{\eta_p \delta} \beta_0^2 \quad (54)$$

The oscillation period (in path length) for a particle is the double of the path length which travel from z_{min} to z_{max} is

$$L(H_0) = \int ds = 2 \int_{z_{min}}^{z_{max}} -\frac{\beta_0^2}{\eta_p \delta(H_0, z)} dz \quad (55)$$

where

$$\delta(H_0, z) = \sqrt{2\beta_0^2 \cdot \frac{1}{\eta_p} \left\{ \frac{qV_{rf}}{\omega_{rf} P_0 C_0} \left[-\cos\left(\phi_s - \frac{\omega_{rf} z}{c}\right) \right] + \sin \phi_s \cdot \frac{\omega_{rf} z}{c} - H_0 \right\}} \quad (56)$$

Then the longitudinal tune

$$\nu_s(H_0) = \frac{C_0}{L(H_0)} \quad (57)$$

Assuming $\phi_s = 0$, (ref S.Y. Lee)

$$\nu_s(\hat{\phi}) \approx \nu_s \left(1 - \frac{\hat{\phi}^2}{16} \right) \quad (58)$$

where $\hat{\phi} = -\omega_{rf} z_{max} / c$.

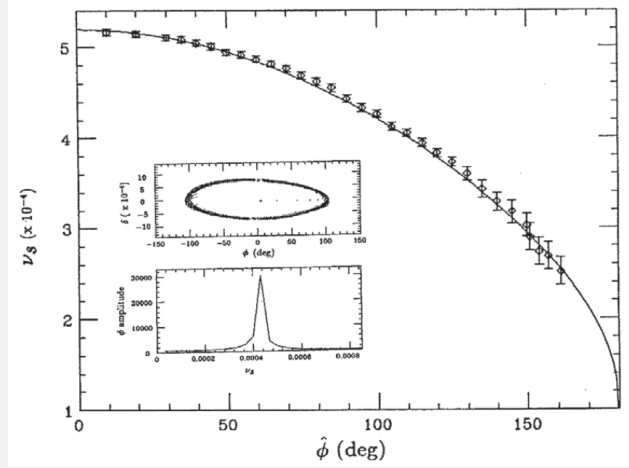


Figure 2: The measured synchrotron tune obtained by taking the FFT of the synchrotron phase coordinate is plotted as a function of the maximum phase amplitude of the synchrotron oscillations. The zero amplitude synchrotron tune was $\nu_s = 5.2 \times 10^{-4}$ (S.Y. Lee)

Small amplitude

$$\begin{aligned} H(z, \delta; s) &= \frac{qV_{rf}}{w_{rf}P_0C_0} \left[-\cos\left(\phi_s - \frac{w_{rf}z}{c}\right) + \sin\phi_s\left(\frac{w_{rf}z}{c}\right) \right] - \frac{1}{2} \frac{1}{\beta_0^2} \eta_p \cdot \delta^2 \\ &\approx \frac{qV_{rf}w_{rf}\cos\phi_s}{2P_0C_0c^2} z^2 - \frac{1}{2} \frac{1}{\beta_0^2} \eta_p \delta^2 \end{aligned} \quad (59)$$

where the static term has been dropped.

We define the beta function

$$\begin{aligned} \beta_z &= \frac{\hat{z}}{\hat{\delta}} = \sqrt{\frac{P_0C_0c^2 \cdot \eta_p}{qV_{rf}w_{rf} \cdot \cos\phi_s}} \\ &= \frac{\eta_p \cdot C_0}{\beta_0 2\pi\nu_s} \end{aligned} \quad (60)$$

Action/ Angle

(ψ_z, J_z) ?

$$J_z = \frac{1}{2\pi} \oint \delta(H_0, z) dz = \frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \delta(H_0, z) dz \quad (61)$$

H_0 is the number of Hamiltonian of the specific particle.

2nd type of generating function is used for the transformation from $(z, \delta) \rightarrow (\phi_z, J_z)$,

$$F_2(z, J_z), \quad \delta = \frac{\partial F_2(z, J_z)}{\partial z}, \quad \phi_z = \frac{\partial F_2(z, J_z)}{\partial J_z} \quad (62)$$

Since we have

$$\delta(H_0, z) = \pm \sqrt{-(H_0 - V(z)) \cdot \frac{2\beta_0^2}{\eta_p}} \Rightarrow \quad (63)$$

$$F_2(z, J_z) = \int_{z_{\min}}^z dz' \cdot \delta(H_0(J_z), z') \quad (64)$$

Then we could get the one-to-one relationship between $(z, \delta) \leftrightarrow (\phi_z, J_z)$.

Don't forget

$$\frac{d\phi_z}{ds} = \frac{\partial H(J_z)}{\partial J_z} \quad (65)$$

Locality of RF Cavity

Recall that map through RF cavity

$$\Delta\delta = \frac{q}{P_0 c} V_{rf} \sin(\phi_s - \omega_{rf} \frac{z}{c}) - \frac{q}{P_0 c} V_{rf} \sin \phi_s \quad (66)$$

Map through arc

$$\Delta z = R_{56} \delta = -\frac{C_0}{\beta^2} \eta_p \delta \quad (67)$$

The linear map around the entrance of the cavity,

$$\begin{aligned} \begin{pmatrix} z \\ \delta \end{pmatrix}_{n+1} &= \begin{pmatrix} 1 & -\frac{C_0}{\beta_0^2} \eta_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{q}{P_0 c} V_{rf} \cos \phi_s \frac{\omega_{rf}}{c} & 1 \end{pmatrix} \begin{pmatrix} z \\ \delta \end{pmatrix}_n \\ &= \begin{pmatrix} 1 + \frac{C_0 \eta_p \omega_{rf} q V_{rf}}{\beta_0^2 c P_0 c} \cos \phi_s & -\frac{C_0}{\beta_0^2} \eta_p \\ -\frac{q}{P_0 c} \frac{\omega_{rf}}{c} V_{rf} \cos \phi_s & 1 \end{pmatrix} \begin{pmatrix} z \\ \delta \end{pmatrix}_n \end{aligned} \quad (68)$$

$$\frac{\text{tr} M}{2} = 1 + \frac{\pi h \eta_p}{2 \beta_0 P_0 c} \cdot q V_{rf} \cos \phi_s \quad (69)$$

The 'exact' tune considering locality of RF cavity is

$$\nu_{s,exact} = \frac{1}{2\pi} \cos^{-1} \left(\frac{\text{Tr} M}{2} \right) = \frac{1}{2\pi} \cos^{-1} \left[1 - \frac{(2\pi \nu_s)^2}{2} \right] \quad (70)$$

The stability condition requires that

$$\nu_s < \frac{1}{\pi} \quad (71)$$

3 Selected topics

3.1 RF Phase Modulation (ref: S.Y. Lee)

Only a sinusoidal RF phase modulation is considered

$$\varphi = a \sin(\nu_m \theta + \chi_0) \quad (72)$$

With

- ν_m : modulation tune
- a : modulation amplitude
- χ_0 : arbitrary phase factor
- θ : orbit angle

The Hamiltonian:

$$H(\phi, \mathcal{P}; \theta) = \underbrace{\frac{1}{2} \nu_s \mathcal{P}^2 + 2 \nu_s \sin^2 \frac{\phi}{2}}_{H_0} + \underbrace{\nu_m a \mathcal{P} \cos(\nu_m \theta + x_0)}_{H_1} \quad (73)$$

ϕ is RF phase experienced by particle. We assume $\phi_s = 0$ and $\eta < 0$.

$$\mathcal{P} \equiv -\frac{h|\eta|}{\nu_s} \delta_p \quad \left(\delta_p \equiv \frac{\Delta P}{P_0} \right) \quad (74)$$

Reminder: $dt \cdot \omega_0 = d\theta$, $\nu_s = \sqrt{\frac{h|\eta|eV_{rf}}{2\pi\beta^2 E}}$

Without Phase Modulation

Action

$$J(H_0) = \frac{1}{2\pi} \oint \mathcal{P}(\phi, H_0) d\phi \quad (75)$$

For small oscillation $J = \frac{1}{2}(\phi^2 + \mathcal{P}^2)$.

Tune

$$\nu_s(J) = \frac{\partial H_0}{\partial J} \approx \nu_s(1 - \frac{J}{8} \dots) \quad (76)$$

For the canonical transformation

$$(\phi, \mathcal{P}) \rightarrow (\psi, J) \quad \text{Here we use } \psi \text{ instead of } \phi_z \quad (77)$$

The generating function

$$F_2(\phi, J) = \int_0^\phi \mathcal{P}(\phi', H_0(J)) d\phi' \quad (78)$$

We could do Fourier expansion for $\mathcal{P}$ in (ψ, J)

$$\mathcal{P} = \sum_n f_n(J) e^{in\psi} \quad (79)$$

and get

$$\mathcal{P}(\psi, J) \approx (2J)^{1/2} \cos \psi + \frac{(2J)^{3/2}}{64} \cos 3\psi \dots \quad (80)$$

Together with (the term in $H(\phi, \mathcal{P})$)

$$2 \sin^2 \left(\frac{\phi}{2} \right) = \sum_n G_n(J) e^{in\psi} \approx -\frac{J}{2} \cos 2\psi - \frac{J^2}{32} \cos 4\psi \dots \quad (81)$$

We get

$$H_0(\psi, J; \theta) \approx \nu_s J - \frac{1}{16} \nu_s J^2 \quad (82)$$

With Modulation in (ψ, J)

The phase modulation term in Hamiltonian

$$H_1 = \nu_m a \sqrt{J/2} [\cos(\psi + \nu_m \theta + \chi_0) + \cos(\psi - \nu_m \theta - \chi_0)] \\ + \nu_m a \frac{(2J)^{3/2}}{128} [\cos(3\psi + \nu_m \theta + \chi_0) + \cos(3\psi - \nu_m \theta - \chi_0)] + \dots, \quad (83)$$

Since $\mathcal{P}$ only contains odd frequency $(\psi, 3\psi, 5\psi, \dots)$ (reminder: $H_1 = \nu_m a \mathcal{P} \cos(\nu_m \theta + \chi_0)$), H_1 only induce odd order parametric resonances:

$$\nu_m : \nu_s = 1 : 1 \quad \text{or} \quad 3 : 1 \quad \dots \quad (84)$$

In the following, we only consider $\nu_m \approx \nu_s$.

$$H(\psi, J; \theta) \approx \nu_s J - \frac{1}{16} \nu_s J^2 + \frac{\nu_s a}{2} J^{1/2} \cos(\psi - \nu_m \theta - \chi_0) \quad (85)$$

In resonance rotating frame

to resonance rotating frame: $(\psi, J) \rightarrow (\chi, I)$ With generating function

$$F_2(\psi, I) = (\psi - \nu_m \theta - \chi_0 - \pi)I, \quad \Rightarrow \quad (86)$$

$$\chi = \psi - \nu_m \theta - \chi_0 - \pi, \quad I = J; \quad (87)$$

And the new Hamiltonian (time independent)

$$\tilde{H}(\chi, I; \theta) = (\nu_s - \nu_m)I - \frac{1}{16}\nu_s I^2 - \nu_s \frac{a}{\sqrt{2}} I^{1/2} \cos \chi. \quad (88)$$

A torus of particle motion will follow a constant Hamiltonian contour.

Fixed points of Hamiltonian:

$$\dot{I} = 0, \quad \dot{\chi} = 0. \quad (\chi = 0 \text{ or } \pi) \quad (89)$$

Define $g = \sqrt{2I} \cdot \cos \chi$ to represent the phase space coordinate of a fixed point.

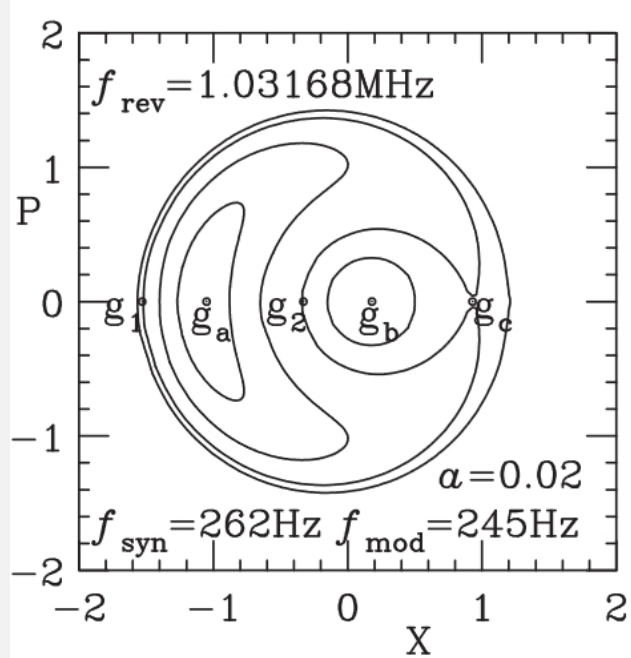


Figure 3: Poincare surfaces with RF phase modulation (SYLee). The SFPs are g_a and g_b , and the UFP is g_c . Here $X \equiv \sqrt{2I} \cos \chi$ and $P \equiv -\sqrt{2I} \sin \chi$

Measurement of RF phase modulation (S.Y. Lee)

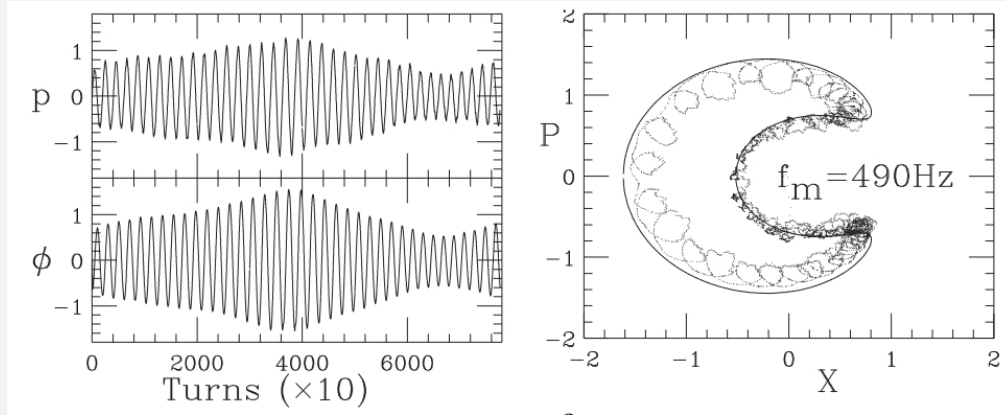


Figure 4: Left plots in normalized off-momentum coordinate $\mathcal{P}$ and the phase ϕ vs revolutions at 10-turn intervals. Right plots: the corresponding Poincare surfaces of section. The solid line shows the Hamiltonian torus. (SYLee) $P \equiv -\sqrt{2I} \sin \chi$

3.2 RF Voltage Modulation (ref: S.Y. Lee)

$\eta < 0$ and $\phi_s = 0$

Normalized phase space coordinates $(\phi, \mathcal{P})$

$$\mathcal{P} \equiv -\frac{h|\eta|}{v_s} \delta_p \quad \left(\delta_p \equiv \frac{\Delta P}{P_0} \right) \quad (90)$$

$$v_s = \sqrt{\frac{h|\eta|eV_{rf}}{2\pi\beta^2 E}} \quad (91)$$

The dynamics equation:

$$\begin{aligned} \frac{d\phi}{d\theta} &= v_s \mathcal{P} \\ \frac{d\mathcal{P}}{d\theta} &= \frac{\eta}{|\eta|} v_s \sin \phi \end{aligned} \quad (92)$$

$$\Rightarrow \ddot{\phi} + v_s^2 \sin \phi = 0 \quad (93)$$

With RF voltage modulation:

$$\Delta V_{rf} = b \cdot V_{rf} \sin(\nu_m \theta + \chi) \quad (94)$$

$$\Rightarrow \ddot{\phi} + v_s^2 [1 + b \sin(\nu_m \theta + \chi)] \sin \phi = 0 \quad (3.114) \quad (95)$$

Linear approximation with $\sin \phi \approx \phi$ (Gennady)

In linear approximation with $\sin \phi = \phi$, reduces to Mathieu equation (frequency is time-dependent),

$$\ddot{\phi} + \nu_s^2 [1 + b \sin(\nu_m \theta + \chi)] \phi = 0 \quad (96)$$

For small b , oscillations become unstable if $\frac{\nu_s}{\nu_m}$ is close to $\frac{n}{2}$ (n is integer),

$$\nu_m = 2\nu_s, \quad \nu_s, \quad \frac{2}{3}\nu_s, \quad \frac{1}{2}\nu_s, \quad \dots \quad (97)$$

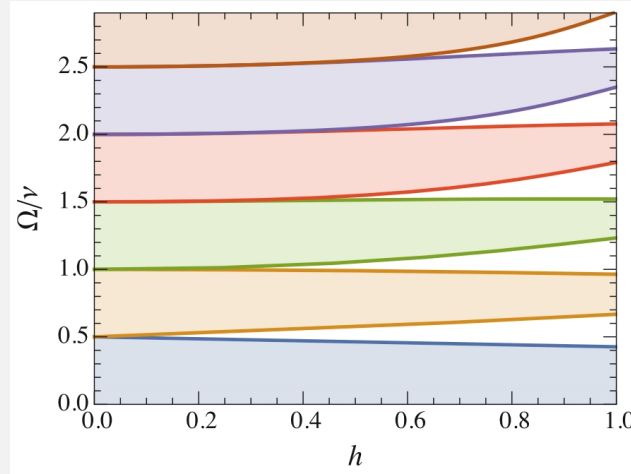


Figure 5: Stability regions for the Mathieu equation as functions of amplitude modulation $h = b$. The stable regions are shadowed. $\Omega/\nu \equiv \nu_m/\nu_s$. (Gennady)

Slow variation of $\omega_0(t)$

$$\ddot{x} + \omega_0^2(t)x = 0 \quad (98)$$

In time ω_0^{-1} , the relative change of ω_0 is small,

$$\omega_0^{-2} \left| \frac{d\omega_0}{dt} \right| \ll 1, \quad (99)$$

a solution as a real part of the complex function $\zeta(t)$ is given by

$$\zeta(t) = A(t) \exp \left(-i \int_0^t \omega_0(t') dt' + i\phi_0 \right). \quad (100)$$

$A(t)$ is the slowly varying amplitude of the oscillation, ϕ_0 is the initial phase. Substituting this into equation,

$$\ddot{A} - 2i\omega_0\dot{A} - i\dot{\omega}_0 A = 0 \quad (101)$$

$\ddot{A}$ is neglected since A is a slow function of time,

$$\begin{aligned} \Rightarrow 2\omega_0\dot{A} + \dot{\omega}_0 A &= 0 \\ \Rightarrow \frac{d}{dt} \ln(A^2\omega_0) &= 0 \\ \Rightarrow A^2\omega_0 &= \text{const} \end{aligned} \quad (102)$$

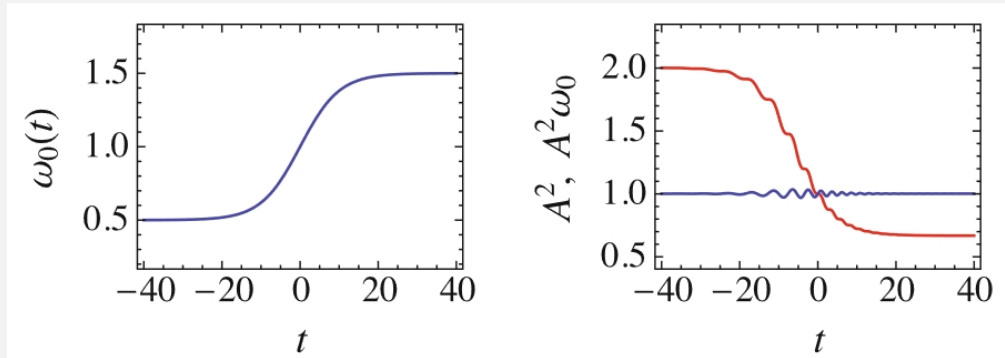


Figure 6: Adiabatic invariance of $A^2\omega_0$: the left panel shows the function $\omega_0(t)$; on the right panel the red curve shows the quantity $x(t)^2 + \dot{x}^2(t)/\omega_0^2(t)$ (which is close to the amplitude squared, A^2) while the blue curve shows the product of this quantity with $\omega_0(t)$ (Gennady).

Parametric resonance due to Voltage Modulation

Perturbed Hamiltonian $H = H_0 + H_1$, with

$$\begin{aligned} H_0 &= \frac{1}{2}v_s \mathcal{P}^2 + v_s(1 - \cos \phi) \\ H_1 &= v_s b \sin(\nu_m \theta + \chi)[1 - \cos \phi] \end{aligned} \quad (103)$$

Fourier expansion of H_1 in action-angle variables (ψ, J) ,

$$H_1(\psi, J; \theta) = v_s b \sum_{n=-\infty}^{\infty} |G_n(J)| \sin(\nu_m \theta - n\psi - \gamma_n) \quad (104)$$

where $\chi = 0$ for simplicity. γ_n is phase of $G_n(J)$.

Recall that

$$2 \sin^2 \left(\frac{\phi}{2} \right) = \sum_n G_n(J) e^{in\psi} \quad (105)$$

Only even n exist,

$$\begin{aligned} G_0 &\approx \frac{1}{2}J + \frac{1}{2048}J^3 + \dots \quad \Rightarrow \Delta v_s = \frac{\partial H_1}{\partial J} \approx \frac{1}{2}v_s b \sin \nu_m \theta \\ G_2 &\approx -\frac{1}{4}J + \frac{1}{128}J^2 + \dots \\ G_4 &\approx -\frac{1}{64}J^2 + \frac{1}{2048}J^3 + \dots \end{aligned} \quad (106)$$

If $\nu_m = n\nu_s$, particle motion can be coherently perturbed by the RF voltage modulation resulting from a resonance driving term. The system is most sensitive to the RF voltage modulation at the second synchrotron harmonic.

$$\nu_m = n\nu_s$$

From (ψ, J) (action-angle coordinates of the unperturbed Hamiltonian) to $(\tilde{\psi}, \tilde{J})$ (resonance rotating frame) using generating function

$$F_2 = \left(\psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n} + \frac{\pi}{2n} \right) \tilde{J} \Rightarrow \quad (107)$$

$$\tilde{\psi} = \psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n} + \frac{\pi}{2n}; \quad \tilde{J} = J. \quad (108)$$

And Hamiltonian $\tilde{H}(\tilde{\psi}, \tilde{J}; \theta) = H_0 + H_1 + \frac{\partial F_2}{\partial \theta}$,

$$\begin{aligned} H_0(\tilde{\phi}, \tilde{J}) &= \frac{1}{2}\nu_s \mathcal{P}^2 + \nu_s(1 - \cos \phi) \\ &\approx \nu_s J - \frac{1}{16}\nu_s J^2 \\ &= \nu_s \tilde{J} - \frac{1}{16}\nu_s \tilde{J}^2 \end{aligned} \quad (109)$$

$$\begin{aligned} H_1(\tilde{\phi}, \tilde{J}; \theta) &= \nu_s b \cdot |G_n(J)| \cdot \sin(\nu_m \theta - n\psi - \gamma_n) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m \theta) \\ &= \nu_s b |G_n(J)| \sin \left[-n\tilde{\psi} + \frac{\pi}{2} \right] + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m \theta) \\ &= \nu_s b |G_n(\tilde{J})| \cdot \cos(n\tilde{\psi}) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m \theta) \end{aligned} \quad (110)$$

$$\frac{\partial F_2}{\partial \theta} = -\frac{\nu_m}{n} \tilde{J} \quad (111)$$

Then we get

$$\begin{aligned} \tilde{H}(\tilde{\psi}, \tilde{J}; \theta) &= H_0 + H_1 + \frac{\partial F_2}{\partial \theta} \\ &= \nu_s \tilde{J} - \frac{1}{16}\nu_s \tilde{J}^2 - \frac{\nu_m}{n} \tilde{J} \\ &\quad + \nu_s b |G_n(\tilde{J})| \cdot \cos(n\tilde{\psi}) + \nu_s b \cdot \frac{J}{2} \cdot \sin(\nu_m \theta) \end{aligned} \quad (112)$$

$$\nu_m \approx 2\nu_s$$

with time average and $n = 2$,

$$\langle \tilde{H}(\tilde{\psi}, \tilde{J}; \theta) \rangle \approx (\nu_s - \frac{\nu_m}{2})\tilde{J} - \frac{\nu_s}{16}\tilde{J}^2 - \frac{\nu_s}{4}b\tilde{J}\cos(2\tilde{\psi}) \quad (113)$$

Now $\langle H \rangle$ is time independent.

Fixed Points:

$$\begin{aligned} \dot{\tilde{J}} &= \frac{\nu_s}{2}b\tilde{J}\sin 2\tilde{\psi} = 0 \\ \dot{\tilde{\psi}} &= \nu_s - \frac{\nu_m}{2} - \frac{\nu_s}{8}\tilde{J} + \frac{\nu_s}{4}b\cos 2\tilde{\psi} = 0 \end{aligned} \quad (114)$$

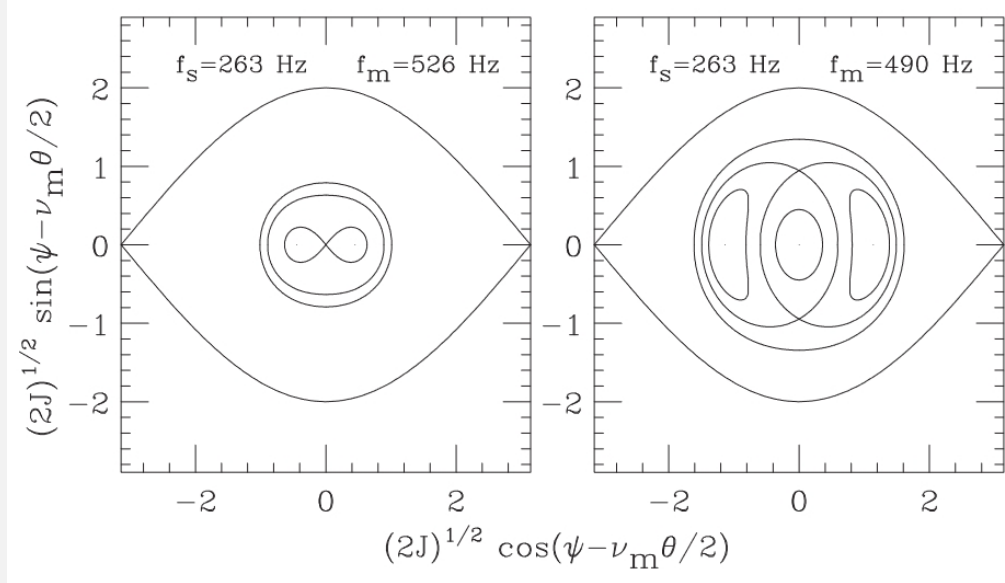


Figure 7: Separatrix and tori of the Hamiltonian in the resonance rotating frame. The voltage modulation amplitude is $b = 0.05$. (S.Y. Lee)

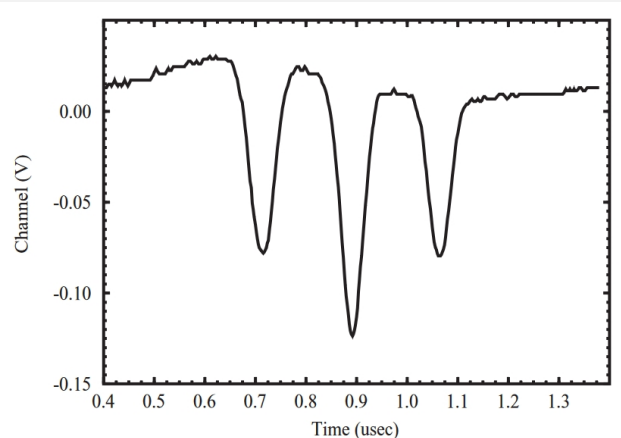


Figure 8: The beam bunch was observed to split into three beamlets in a single rf bucket. The voltage modulation amplitude is $b = 0.05$ at modulation frequency $f_m = 480$ Hz with synchrotron tune $f_s = 263$ Hz. Note that the outer two beamlets rotated around the center beamlet at a frequency equal to half the modulation frequency. (S.Y. Lee)

3.3 Harmonic Cavity

Higher Harmonic cavity $\Rightarrow$
 Flatten the potential well $\Rightarrow$
 Bunch Lengthening

- weaken space charge
- increase IBS time in light source
- mitigate X-Z instability in e+e- collider

For single RF system (orbit angle θ as time),

$$\dot{\delta} = \frac{eV_{rf}}{2\pi\beta^2E} [\sin \phi - \sin \phi_s] \quad (115)$$

- ϕ is the phase coordinate relative to the primary RF cavity
- ϕ_s is the phase angle for the synchronous particle
- V_{rf} is the primary rf cavity voltage

For double RF system

$$\dot{\delta} = \frac{e}{2\pi\beta^2E} \left\{ V_1 [\sin \phi - \sin \phi_{1s}] + V_2 \left[\sin(\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})) - \sin \phi_{2s} \right] \right\} \quad (116)$$

$$\dot{\phi} = h_1 \eta \delta \quad (117)$$

The Hamiltonian

$$H(\phi, \delta; \theta) = \frac{1}{2} h_1 \eta \delta^2 + \frac{e}{2\pi\beta^2E} \left\{ V_1 [\cos \phi - \cos \phi_{1s} + (\phi - \phi_{1s}) \cdot \sin \phi_{1s}] \right. \\ \left. + \frac{h_1}{h_2} V_2 \left[\cos(\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})) - \cos \phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s}) \cdot \sin \phi_{2s} \right] \right\} \quad (118)$$

With $h = \frac{h_2}{h_1}$, $r = \frac{V_2}{V_1}$ and in normalized momentum $\mathcal{P} = -\frac{h_1|\eta|}{v_s}\delta$ ($v_s = \sqrt{\frac{h_1 e V_1 |\eta|}{2\pi\beta^2 E_0}}$, assuming $\eta < 0$),

$$H(\phi, \mathcal{P}; \theta) = \frac{1}{2}v_s \mathcal{P}^2 + v_s \cdot \left\{ (\cos \phi_{1s} - \cos \phi) + (\phi_{1s} - \phi) \sin \phi_{1s} \right. \\ \left. + \frac{r}{h} \cdot [\cos \phi_{2s} - \cos(\phi_{2s} + h(\phi - \phi_{1s})) - h(\phi - \phi_{1s}) \sin \phi_{2s}] \right\} \quad (119)$$

A flattened potential well requires

$$\frac{\partial H}{\partial \phi} = v_s \left\{ \sin \phi - \sin \phi_{1s} + \frac{r}{h} \cdot [h \sin(\phi_{2s} + h(\phi - \phi_{1s})) - h \sin \phi_{2s}] \right\} = 0 \quad (120)$$

$$\Rightarrow \phi = \phi_{1s} \quad (121)$$

$$\frac{\partial^2 H}{\partial \phi^2} = v_s \{ \cos \phi + rh \cos(\phi_{2s} + h(\phi - \phi_{1s})) \} = 0 \quad (122)$$

$$\Rightarrow \cos \phi_{1s} + rh \cos \phi_{2s} = 0 \quad (123)$$

$$\frac{\partial^3 H}{\partial \phi^3} = v_s \cdot \{ -\sin \phi - rh^2 \sin(\phi_{2s} + h(\phi - \phi_{1s})) \} = 0 \quad (124)$$

$$\Rightarrow \sin \phi_{1s} + r \cdot h^2 \cdot \sin \phi_{2s} = 0 \quad (125)$$

Example of harmonic cavity configuration

$h = 2$ and $\phi_{1s} = \phi_{2s} = 0^\circ$

$$H(\phi, \mathcal{P}; \theta) = \frac{v_s}{2} \mathcal{P}^2 + v_s \left\{ (1 - \cos \phi) + \frac{r}{2} [1 - \cos 2\phi] \right\} \quad (126)$$

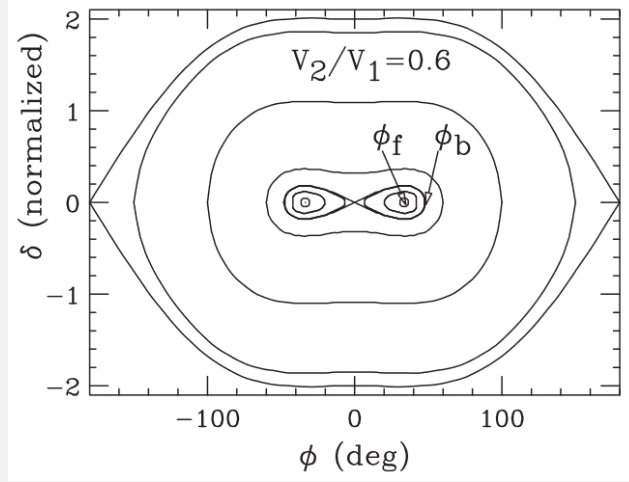


Figure 9: RF bucket and phase space ellipses for a double rf system with $h_2/h_1 = 2$ and $r = V_2/V_1 = 0.6$. (S.Y. Lee)

3.4 Nonlinear Phase Slip Factor

With

$$\eta = \eta_0 + \eta_1 \delta + \dots \quad (127)$$

$$\frac{d\phi}{d\theta} = -\frac{\partial H}{\partial \delta} = h \cdot \eta(\delta) \cdot \delta = h\eta_0\delta + h\eta_1\delta^2 \quad (128)$$

Hamiltonian

$$H(\phi, \delta; \theta) = \frac{1}{2}h \left(\eta_0 + \frac{2}{3}\eta_1\delta \right) \delta^2 + \frac{eV}{2\pi\beta^2E} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s] \quad (129)$$

Fixed Points:

$$\begin{aligned} (\phi, \delta)_{SFP} &= (\phi_s, 0), & (\pi - \phi_s, -\eta_0/\eta_1) \\ (\phi, \delta)_{UFP} &= (\pi - \phi_s, 0), & (\phi_s, -\eta_0/\eta_1) \end{aligned} \quad (130)$$

With $\nu_s = \sqrt{\frac{heV|\eta_0|}{2\pi\beta^2 E_0}}$,

in normalized phase space coordinates ϕ and $\mathcal{P} = \frac{h|\eta_0|}{\nu_s}\delta$ (assuming $\eta_0 > 0, \eta_1 > 0$),

Hamiltonian

$$H(\phi, \mathcal{P}; \theta) = \frac{1}{2}\nu_s \mathcal{P}^2 + \frac{1}{2y}\nu_s \mathcal{P}^3 + \nu_s [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s] \quad (131)$$

where $y \equiv \frac{3h\eta_0^2}{2\eta_1\nu_s}$.

If $|y| \gg 1$, nonlinear phase slip factor is not important, otherwise phase space tori will be deformed.

Two separatrix will cross if

$$H(\phi = \pi - \phi_s, \mathcal{P} = 0) = H(\phi = \phi_s, \mathcal{P} = -\frac{h|\eta_0|}{\nu_s} \frac{\eta_0}{\eta_1}) \quad (132)$$

where $y = y_{cr}$,

$$y_{cr} = \sqrt{27 \left[\left(\frac{\pi}{2} - \phi_s \right) \sin \phi_s - \cos \phi_s \right]} \quad (133)$$

Example of non-linear phase slip factor.

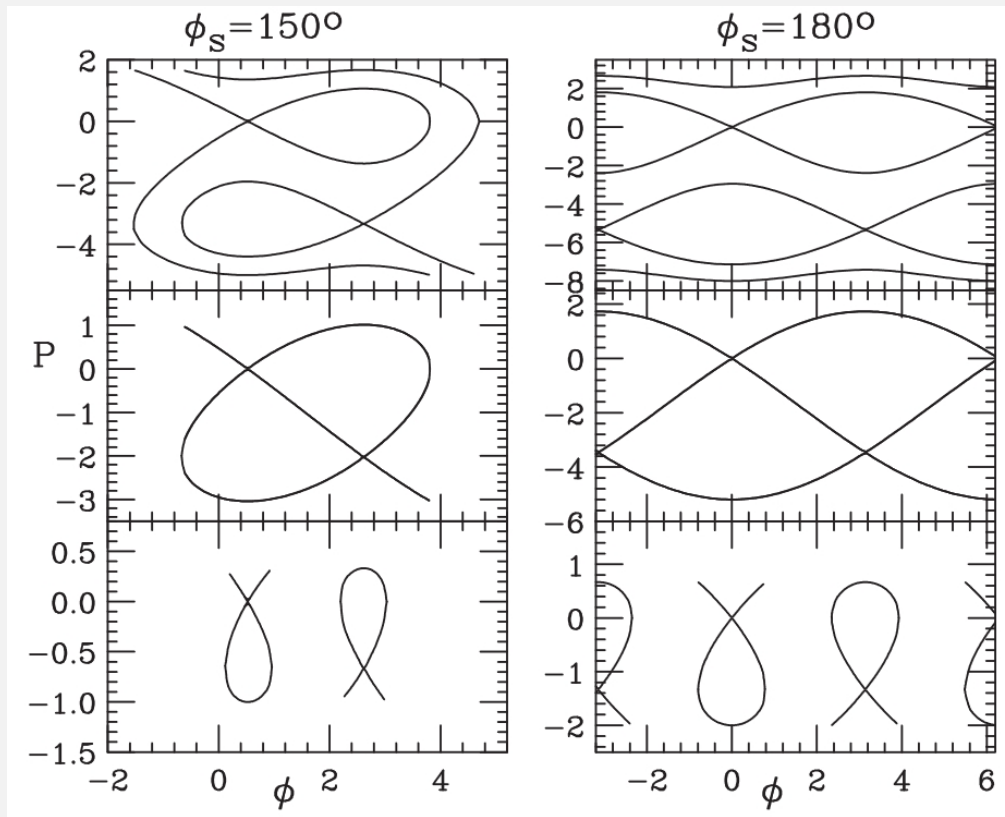


Figure 10: Separatrix in the normalized phase space. Left: $\phi_s = 150^\circ$. $y = 5$ (top), $y = y_{cr} = 3.0406$ (middle) and 1 (bottom). Right: $\phi_s = 150^\circ$. $y = 8$ (top), $y = y_{cr} = 5.1962$ (middle), and 3 (bottom). (S.Y. Lee)

3.5 Local Phase Slippage (ref: Deng, PRAB 24, 094001 (2021))

One turn map in Courant-Snyder Formalism

$$M_z = \begin{pmatrix} \cos \Phi_z + \alpha_z \sin \Phi_z & +\beta_z \sin \Phi_z \\ -\gamma_z \sin \Phi_z & \cos \Phi_z + \alpha_z \sin \Phi_z \end{pmatrix} \quad (134)$$

where $\Phi_z = 2\pi\nu_s$.

The second moments

$$\Sigma_z = \begin{pmatrix} \langle z^2 \rangle & \langle z\delta \rangle \\ \langle z\delta \rangle & \langle \delta^2 \rangle \end{pmatrix} = \begin{pmatrix} \epsilon_z \beta_z + \epsilon_x \mathcal{H}_x & -\epsilon_z \alpha_z \\ -\epsilon_z \alpha_z & \epsilon_z \gamma_z \end{pmatrix} \quad (135)$$

ϵ_z is the longitudinal emittance.

$$\mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D_x'^2 \quad (136)$$

At the middle of RF cavity, the one-turn map (assume no dispersion at RF)

$$M = \begin{pmatrix} 1 & 0 \\ \frac{k}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\eta C_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{k}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{k}{2}\eta C_0 & -\eta C_0 \\ k - (\frac{k}{2})^2 \eta C_0 & 1 - \frac{k}{2}\eta C_0 \end{pmatrix} \quad (137)$$

where $k = \frac{eV_{rf}}{\beta^2 E_0} \cdot \cos \phi_s \cdot \frac{2\pi h}{C_0}$ (h : harmonic number, C_0 : circumference)

In the conventional case,

$$\Phi_z \approx \begin{cases} -\sqrt{k\eta C_0} & , \text{ if } \eta > 0 \\ \sqrt{k\eta C_0} & , \text{ if } \eta < 0 \end{cases} \quad (138)$$

$$\beta_z = \frac{M_{12}}{\sin \Phi_z} \approx \sqrt{\frac{\eta C_0}{k}} = \frac{\eta C_0}{\Phi_z} \quad (139)$$

if $|\eta|$ is small, partial phase slippage may be larger than global.
variation of β_z
partial R_{56} from s_1 to s_2 ,

$$F(s_1, s_2) = - \int_{s_1}^{s_2} \left(\frac{D_x(s)}{\rho(s)} - \frac{1}{\gamma^2} \right) ds \quad (140)$$

at position s_j ,

$$F(s_j, s_{rf-}) + F(s_{rf+}, s_j) = -\eta C_0. \quad (141)$$

The transfer matrix

$$\begin{aligned} T(s_j, s_{rf-}) &= \begin{pmatrix} 1 & F(s_j, s_{rf-}) \\ 0 & 1 \end{pmatrix} \\ T(s_{rf-}, s_{rf+}) &= \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \\ T(s_{rf+}, s_j) &= \begin{pmatrix} 1 & F(s_{rf+}, s_j) \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (142)$$

One turn map at s_j ,

$$M(s_j) = T(s_{rf+}, s_j) T(s_{rf-}, s_{rf+}) T(s_j, s_{rf-}) \quad (143)$$

$$\beta_z(s_j) = \frac{M_{12}(s_j)}{\sin \Phi_z} = \frac{-\eta C_0 + F(s_{rf+}, s_j) \cdot F(s_j, s_{rf-}) \cdot k}{\sin \Phi_z} \quad (144)$$

More exact longitudinal emittance:

$$\epsilon_z = \frac{5}{96\sqrt{3}} \frac{\alpha_F \cdot \lambda_e^2 \cdot \gamma^5}{\alpha_L} \oint \frac{\beta_z(s)}{|\rho(s)|^3} ds \quad (145)$$

where $\alpha_F = 1/137$ the fine structure constant, $\lambda_e = \lambda/2\pi = 386$ fm the reduced Compton wavelength of electron, $\alpha_L = J_z U_0 / 2E_0$ the longitudinal damping constant (J_z Longitudinal damping partition).

Example of local β_z and partial phase slip.

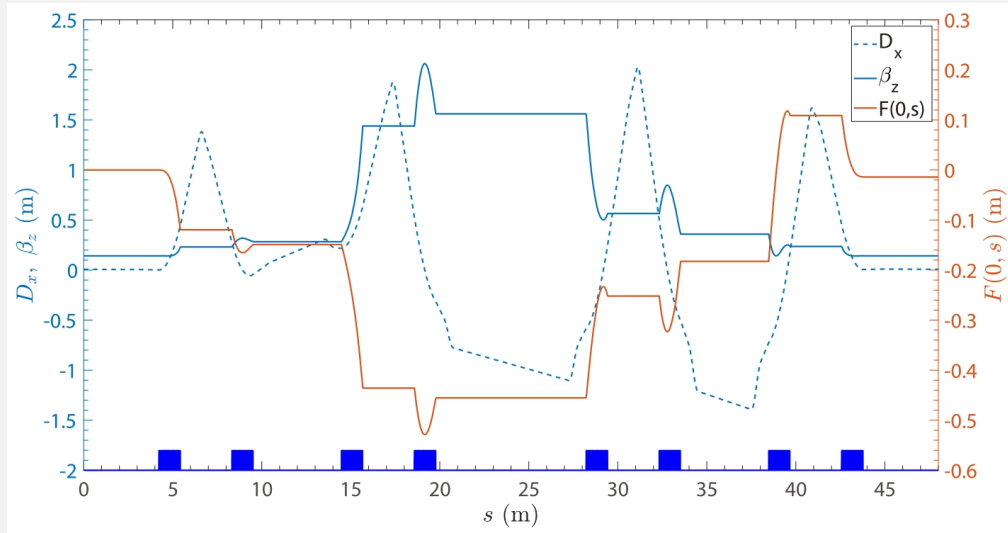


Figure 11: The horizontal dispersion D_x , longitudinal beta function β_z , and $F(0, s)$ of the MLS lattice. In this plot, the zero-length rf is placed at $s_{rf} = 0$ m and $V_{rf} = 80$ MV is applied. (X.J. Deng)

3.6 Synchrotron Radiation

With SR energy loss

$$U_\delta = U_0(1 + \delta)^3 \quad (146)$$

The dynamics equation becomes

$$\begin{aligned} \frac{dz}{dt} &= -c\eta\delta \\ \frac{d\delta}{dt} &= \frac{1}{E_0 T_0} [eV(z) - U_0(1 + \delta)^3] \end{aligned} \quad (147)$$

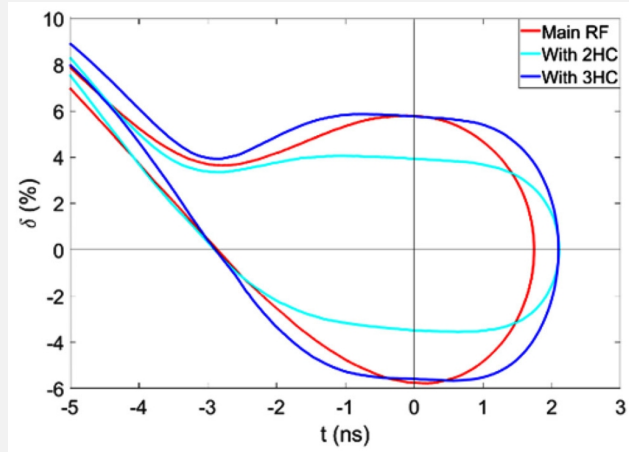


Figure 12: Longitudinal acceptance obtained by ELEGANT (NST (2019) 30:113)

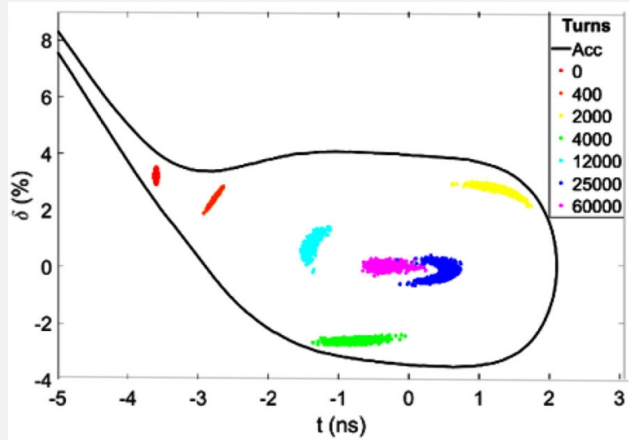


Figure 13: Longitudinal distribution during injection (NST (2019) 30:113)

3.7 Adiabatic Damping

Now $E_s, \beta_s, \omega_0, \phi_s, V_0$ and η depends on time t ,

$$\begin{aligned} (\text{Ampl. of } \tau) &\propto \left(\frac{-\eta}{\beta_s^2 E_s \omega_0^2 e V_0 \cos \phi_s} \right)^{\frac{1}{4}}, \\ (\text{Ampl. of } E - E_s) &\propto \left(\frac{-\eta}{\beta_s^2 E_s \omega_0^2 e V_0 \cos \phi_s} \right)^{-\frac{1}{4}}. \end{aligned} \quad (148)$$

The emittance in $(\tau, \Delta E)$ will keep unchanged.

reminder: $\Delta\phi = \omega_{rf}\tau$

In phase space coordinates

$$\begin{aligned} z &\equiv s - \tau \beta_s c \\ \delta_p &\equiv \frac{1}{\beta_s^2} \frac{\Delta E}{E_s} \end{aligned} \quad (149)$$

we have

$$(\text{Ampl. of } z) \times (\text{Ampl. of } \delta_p) \propto \frac{1}{\beta_s E_s} \times (\text{Ampl. of } \tau) \times (\text{Ampl. of } \Delta E) \quad (150)$$

In phase space coordinates

$$\begin{aligned} z &\equiv \frac{s}{\beta_s} - c\tau \\ \delta &\equiv \frac{\Delta E}{P_s c} \end{aligned} \quad (151)$$

we have

$$(\text{Ampl. of } z) \times (\text{Ampl. of } \delta) \propto \frac{1}{\beta_s E_s} \times (\text{Ampl. of } \tau) \times (\text{Ampl. of } \Delta E) \quad (152)$$

3.8 RF cavity at dispersion location

One turn arc map (only x and z) around RF cavity

$$M_{arc} = H^{-1} \begin{pmatrix} \cos \mu_x + \alpha \sin \mu_x & \beta_x \sin \mu_x & 0 & 0 \\ -\gamma_x \sin \mu_x & \cos \mu_x - \alpha_x \sin \mu_x & 0 & 0 \\ 0 & 0 & 1 & -\eta C \\ 0 & 0 & 0 & 1 \end{pmatrix} H \quad (153)$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 & -D \\ 0 & 1 & 0 & -D' \\ D' & -D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (154)$$

The map through the RF cavity

$$M_{cav} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{pmatrix} \quad (155)$$

where $k = -\frac{qV_{rf}rw_{rf}}{P_0c^2} \cos \phi_s$

The one turn map is $M = M_{arc}M_{cav}$.

The eigenvalues of M are $\lambda_1, 1/\lambda_1, \lambda_2, 1/\lambda_2$, we define

$$s = \lambda_1 + 1/\lambda_1 \text{ or } \lambda_2 + 1/\lambda_2 \quad (156)$$

s is the root of the following equation

$$s^2 - [2 \cos \mu_x + 2 \cos \mu_s + \mathcal{H}_x k \sin \mu_x] \cdot s + [4 \cos \mu_x \cos \mu_s + 2 \mathcal{H}_x k \sin \mu_x] = 0 \quad (157)$$

Don't forget

$$2 \cos \mu_s = 2 - k\eta C \quad (158)$$

Reminder: $\mathcal{H}_x = \gamma_x D^2 + 2\alpha_x D D' + \beta D'^2$ which ever appear in the diffusion term in the evaluation of horizontal emittance due to SR.

It could be obtained that

$$s = \left(\cos \mu_x + \cos \mu_s + \frac{1}{2} \mathcal{H}_x k \sin \mu_x \right) \pm \sqrt{\left(\cos \mu_x + \cos \mu_s + \frac{1}{2} \mathcal{H}_x k \sin \mu_x \right)^2 - (4 \cos \mu_x \cos \mu_s + 2 \mathcal{H}_x k \sin \mu_x)} \quad (159)$$

Consider the case, $\eta > 0$, we have $k > 0$. When μ_x above integer, s would be complex (unstable) if $\cos \mu_x \approx \cos \mu_s$.

3.9 Crab Cavity

The map of crab cavity (only consider x and z)

$$M_{\text{crab}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_{\text{crab}} & 0 \\ 0 & 0 & 1 & 0 \\ k_{\text{crab}} & 0 & 0 & 1 \end{pmatrix} \quad (160)$$

where $k_{\text{crab}} = \frac{eV_{\perp,rf}\omega_{\perp,rf}}{cE_0}$. The crab RF voltage

$$V_{\perp}(t) = V_{\perp,rf} \sin(\omega_{\perp,rf}t) \approx V_{\perp,rf}\omega_{\perp,rf}\frac{z}{c} \quad (161)$$

The one turn map is

$$M_{\text{quadrif}} = M_{\text{arc}} \cdot M_{\text{rf}} \cdot M_{\text{crab}} \quad (162)$$

Here we assume no dispersion at cavity.

$e^{\pm i\mu}$ is the eigenvalues of M

$$2 \cos \mu = \cos \mu_s + \cos \mu_x \pm \sqrt{(\cos \mu_s - \cos \mu_x)^2 - \beta_x C_0 \eta k_{\text{crab}}^2 \sin \mu_x} \quad (163)$$

4 Spectrum

- Fourier transforms:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \quad (164)$$

In this equation $j = \sqrt{-1}$ and ω is the angular frequency

- The inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \quad (165)$$

- The image (wall) current i_W has a line density equal to the line density of the beam current i_b ,

$$i_W(t) = -i_b(t) \text{ and } I_W(\omega) = -I_b(\omega) \quad (166)$$

with an overall minus sign since it is an image current.

A single particle with constant revolution frequency

- The current of a single, unit-charge particle with a constant revolution frequency ω_r , is a periodic set of impulses spaced a time $T = 2\pi/\omega_r$ apart

$$i_b(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad (167)$$

- Fourier transforming gives

$$\begin{aligned} I_b(\omega) &= \int_{-\infty}^{\infty} i_b(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} e^{-j\omega nT} = \sum_{n=-\infty}^{\infty} e^{-j2\pi n\omega/\omega_r} \\ &= \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \end{aligned} \quad (168)$$

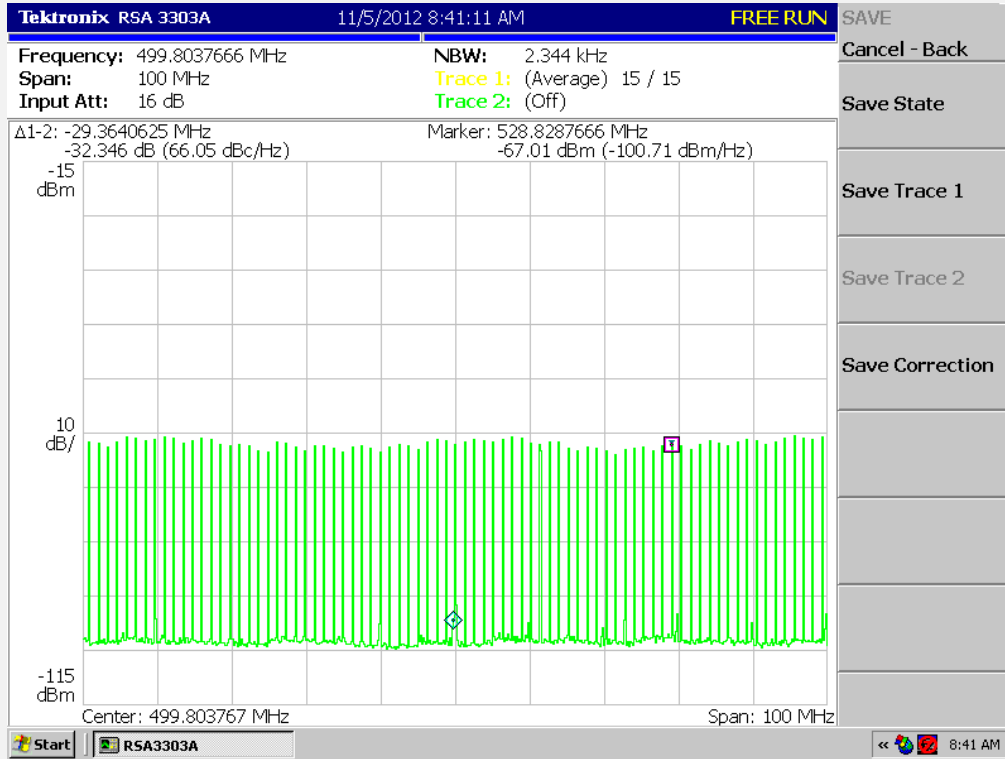


Figure 14: Single bunch spectrum obtained by spectrum analyzer (BEPCII).

Frequency Modulation By Synchrotron Oscillations

- Synchrotron motion modulates the arrival time by

$$\tau = \tau_a \cos(\omega_s t + \psi) \quad (169)$$

where τ_a is the synchrotron oscillation amplitude, ω_s is the synchrotron frequency and ψ is the phase.

- The current is

$$i_b(t) = \sum_{n=-\infty}^{\infty} \delta(t - [nT + \tau_a \cos(\omega_s nT + \psi)])$$

- The Fourier transform is

$$I_b(\omega) = \sum_{n=-\infty}^{\infty} \exp(-j\omega[nT + \tau_a \cos(\omega_s nT + \psi)]) \quad (170)$$

- With a Bessel function expansion

$$e^{-jz \cos \theta} = \sum_{k=-\infty}^{\infty} J_k(z) e^{jk(\theta - \pi/2)} \quad (171)$$

The Fourier transform can be rewritten as

$$\begin{aligned} I_b(\omega) &= \sum_{n,k=-\infty}^{\infty} J_k(\omega \tau_a) e^{jk(\omega_s nT + \psi - \pi/2)} e^{-jn\omega T} \\ &= \sum_{n,k=-\infty}^{\infty} e^{jk(\psi - \pi/2)} J_k(\omega \tau_a) e^{-j2\pi n(\omega - k\omega_s)/\omega_r} \\ &= \omega_r \sum_{k=-\infty}^{\infty} e^{jk(\psi - \pi/2)} J_k(\omega \tau_a) \sum_{n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \end{aligned} \quad (172)$$

- For each rotation harmonic there is an infinite number of sidebands. They are displaced from the rotation harmonic by $k\omega_s$, $k = -\infty, \dots, \infty$ and have different envelopes. The first maximum of J_k is at $\omega\tau_a \approx k$. When $|\omega| \ll 1/\tau_a$ only the rotation harmonics, $k = 0$, are present, and as the frequency increases more sidebands appear.

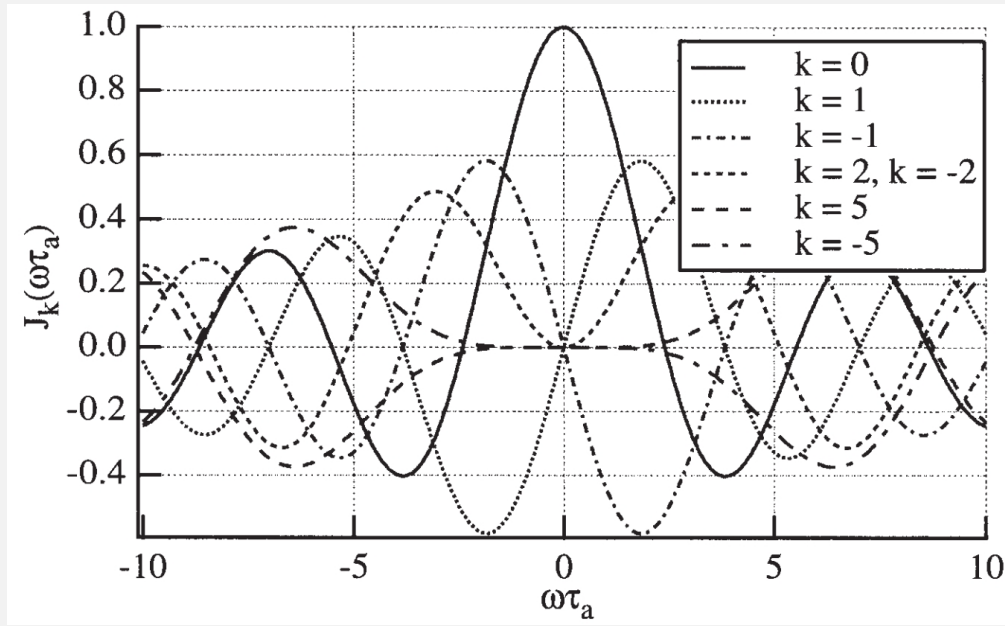


Figure 15: Envelope of synchrotron sideband.

Multiple Particles

With the longitudinal phase space density of the beam $\rho(\tau_a, \psi)$, where $\rho\tau_a d\tau_a d\psi$ is the charge in phase space area $\tau_a d\tau_a d\psi$, the longitudinal spectrum of a beam is

$$I(\omega) = \omega_r \sum_{k,n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \int_0^{2\pi} d\psi \rho(\tau_a, \psi) e^{jk\psi} J_k(\omega\tau_a) \quad (173)$$

If phase space density is $\rho(\tau_a, \psi) = \rho_0(\tau_a)/2\pi$, the beam signal is

$$\begin{aligned} I(\omega) &= \omega_r \sum_{k,n=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \frac{1}{2\pi} \int_0^{2\pi} d\psi \rho_0(\tau_a) e^{jk\psi} J_k(\omega\tau_a) \\ &= \omega_r \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \rho_0(\tau_a) J_0(\omega\tau_a) \end{aligned} \quad (174)$$

If the beam has charge Q and is Gaussian in τ with rms bunch length σ_τ ,

$$\rho_0(\tau_a) = \frac{Q}{\sigma_\tau^2} \exp(-\tau_a^2/2\sigma_\tau^2) \quad (175)$$

and

$$I(\omega) = Q\omega_r \exp(-\omega^2\sigma_\tau^2/2) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) \quad (176)$$

The appearance of synchrotron sidebands and azimuthal structure in longitudinal phase space are directly related.

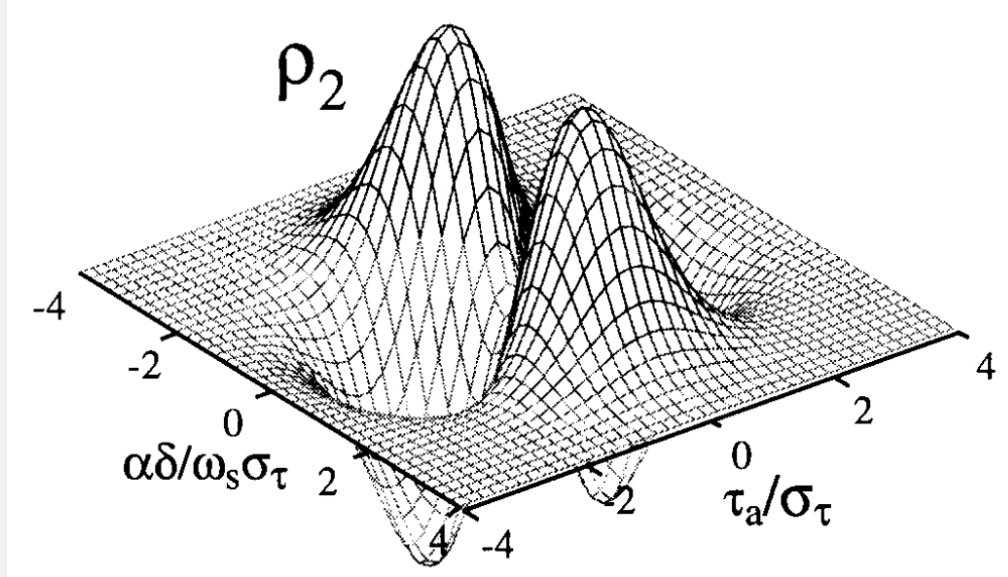
The phase space density can be written as a Fourier expansion

$$\rho(\tau_a, \psi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \rho_m(\tau_a) e^{im\psi} \quad (177)$$

Substituting into eq. (173),

$$\begin{aligned} I(\omega) &= \frac{\omega_r}{2\pi} \sum_{k,n,m=-\infty}^{\infty} \delta(\omega - k\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \int_0^{2\pi} d\psi \rho_m(\tau_a) e^{j(k+m)\psi} J_k(\omega\tau_a) \\ &= \omega_r \sum_{m,n=-\infty}^{\infty} \delta(\omega + m\omega_s - n\omega_r) \int_0^{\infty} \tau_a d\tau_a \rho_m(\tau_a) J_k(\omega\tau_a) \end{aligned} \quad (178)$$

As a specific example, suppose that the beam has a quadrupole perturbation shown in the following figure.



$$\rho(\tau_a, \psi) = \frac{\rho_0(\tau_a)}{2\pi} \left(1 + A_2 \frac{\tau_a^2}{\sigma_\tau^2} \cos(2\psi) \right) \quad (179)$$

where ρ_0 is given by eq. (175). Substituting into (178) and performing the integrals

$$I(\omega) = Q\omega_r \exp(-\omega^2\sigma_\tau^2/2) \left[\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r) + A_2 \frac{\omega^2\sigma_\tau^2}{2} \left(\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r + 2\omega_s) + \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_r - 2\omega_s) \right) \right] \quad (180)$$

The envelopes are shown in the following figure with $A_2 = 0.01$.

