An Introduction to Accelerators

Part III

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Outline

1. Principles of Accelerators
2. Contributions to Science and Other Applications
3. A Simple Introduction of Accelerator Physics

3. A Simple Introduction of Accelerator Physics

References:

- A.W.Chao, Lectures on Accelerator Physics, World Scientific, [2020)
- N.Q. Liu, Accelerator Theory, 2nd edition, Tsinghua University Press, [2004]

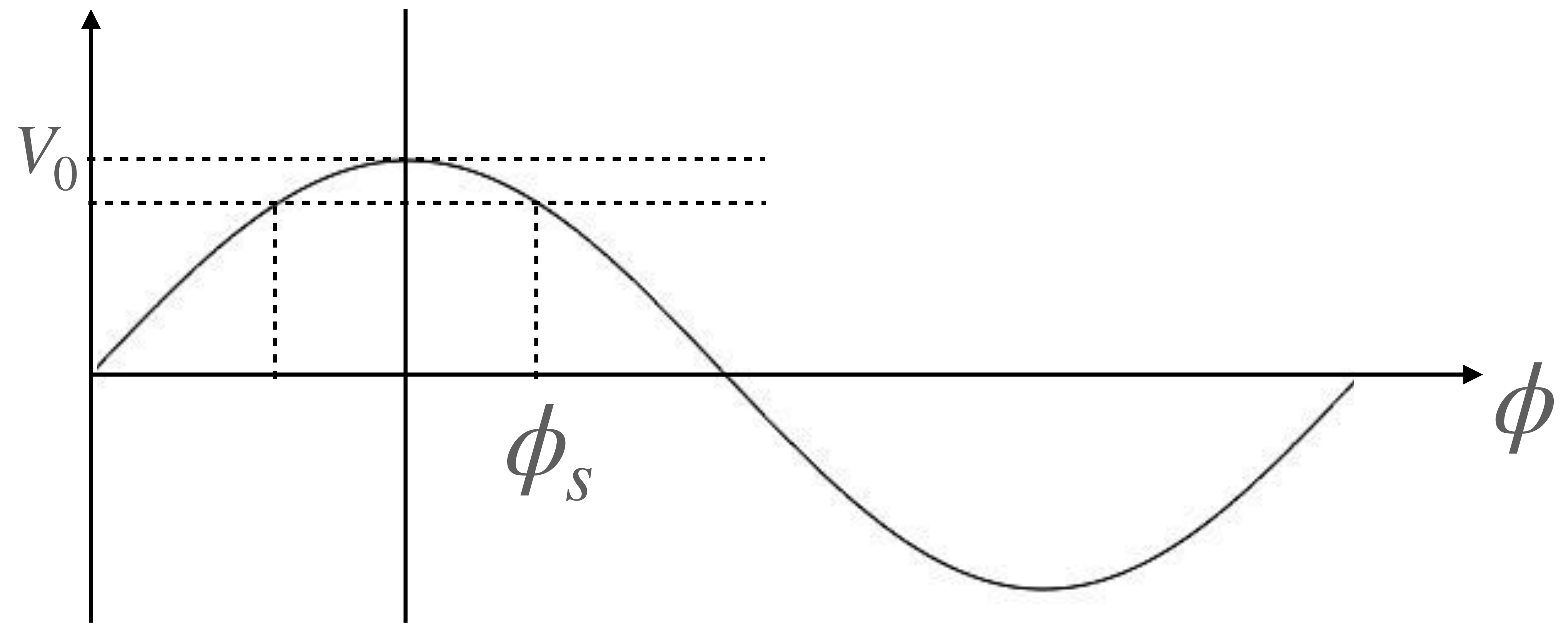
Accelerator physics is about the laws of charged particles moving in the electromagnetic fields of the accelerator machine and the charged particles themselves. Two aspects of the physics are most important:

- **Stability**: How to keep a particle be accelerated continuously and stably on its orbit? -**The highest energy**.
  - ☑ Longitudinal dynamics: the principle of phase stability
  - ☑ Transverse dynamics: the alternative gradient focusing and its stability
- **Instability**: How to keep as many as particles be accelerated? -**The threshold current**.
  - ☑ The collective effects

2.1 Longitudinal dynamics

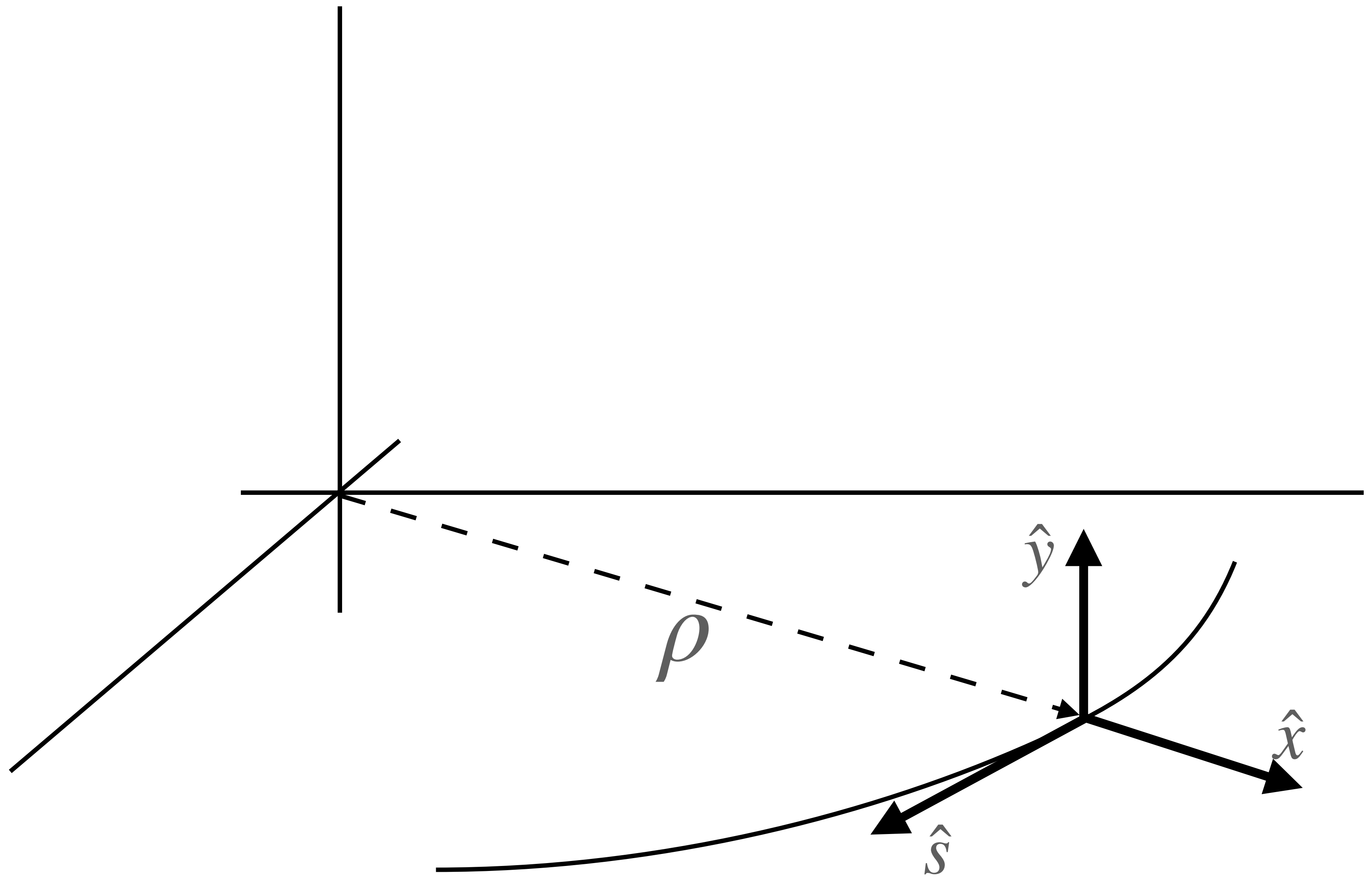
The movement along the designed trajectory s is the longitudinal motion of the particle.

The voltage of the particle experience traversing the rf cavity is $V(t) = V_0 \sin(\omega_{rf} t + \phi_s)$, with $\omega_{rf} = h\omega_0$

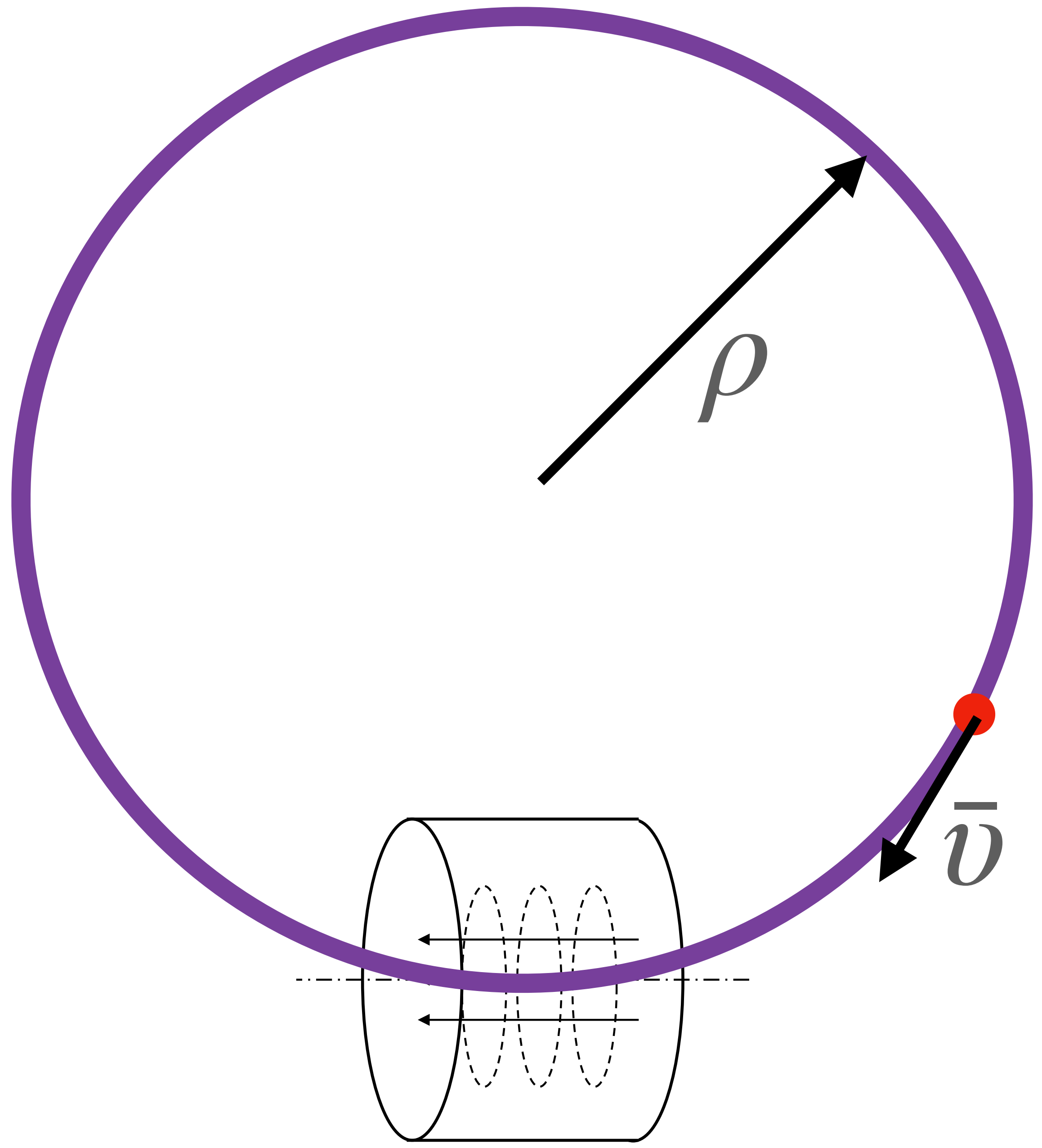


The **synchronous phase** ϕ_s id defined as the RF phase seen by the **idealized synchronous particle** as it traverse the cavity. This means the accelerating voltage seen by the synchronous particle, turn by turn, is

$$V_s = V_0 \sin(\phi_s)$$



The designed trajectory



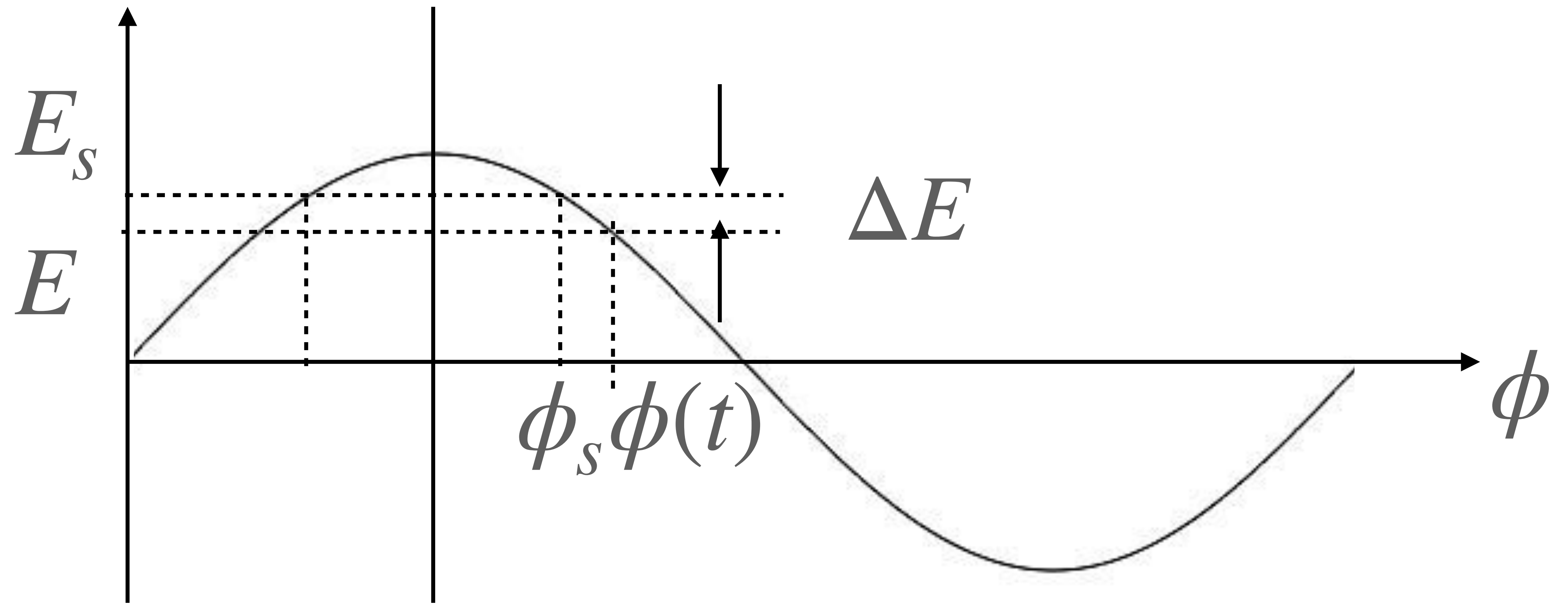
The longitudinal coordinates: ($\phi, \Delta E$) and (z, δ)

energy of of a particle relative to the synchronous particle,

$$\Delta E = E - E_s$$

and the RF phase at the arrival time of the particle ϕ

$$\delta = \frac{\Delta P}{P_s} = \frac{1}{\beta_s^2}\frac{\Delta E}{E_s}$$

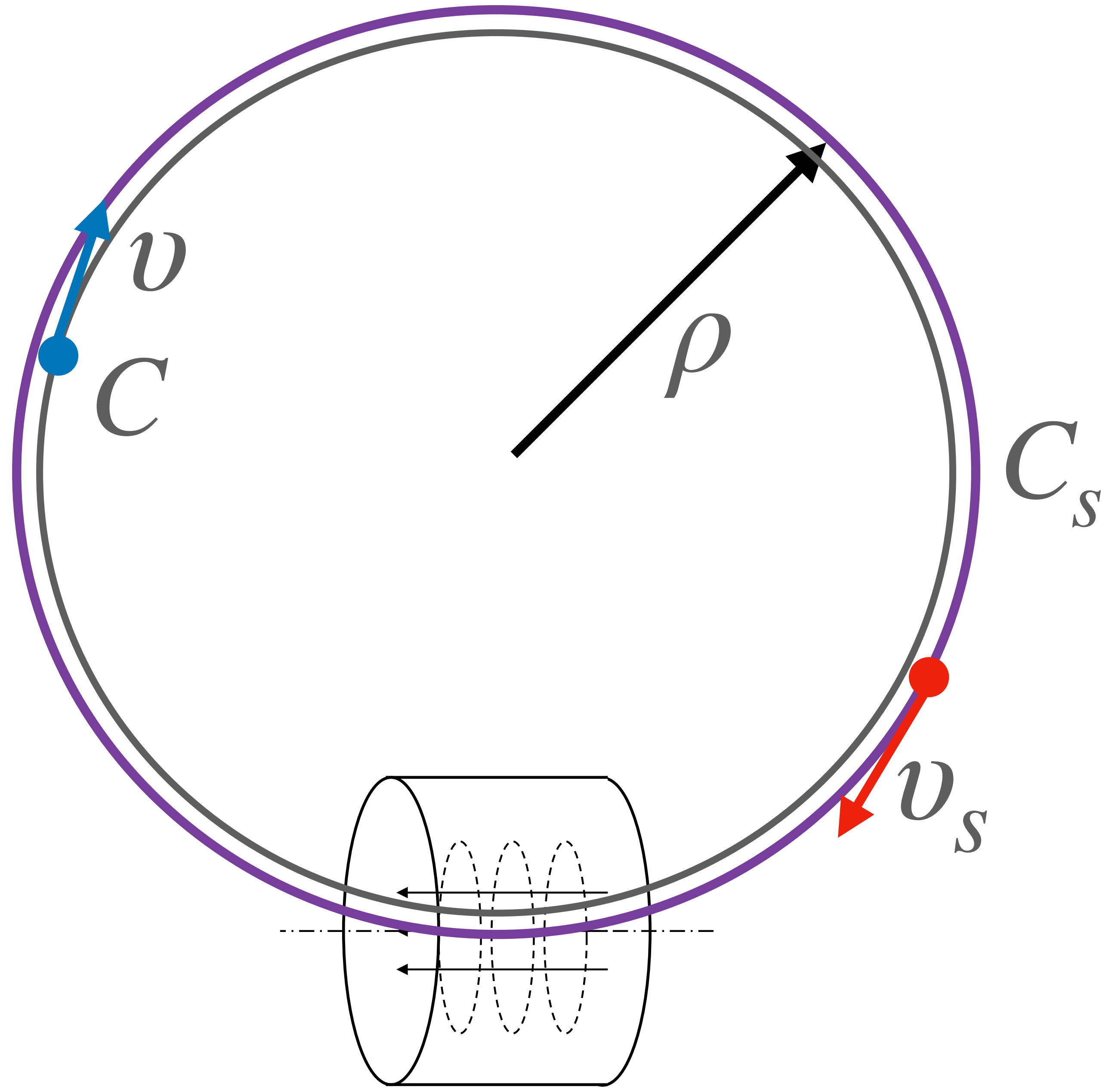


Energy variation:

In the adiabatic approximation, the rate of change of ΔE, due to the RF acceleration is given by

$$\dot{\Delta E} = eV_0\frac{\omega_0}{2\pi}(\sin\phi - \sin\phi_s)$$

$$\dot{\delta} = eV_0\frac{\omega_0}{2\pi\beta_s^2 E_s}(\sin\phi - \sin\phi_s)$$



Phase variation:

For $T = \dfrac{C_s + \Delta C}{v_s + \Delta v}$, $T_s = \dfrac{C_s}{v_s}$, with Taylor expansion and

keeping only the 1st order of $\dfrac{\Delta C}{C_s}$ and $\dfrac{\Delta v}{v_s}$,

$$\dot{\phi} = \frac{\Delta\phi}{T} = \omega_{rf}\frac{T - T_s}{T} \approx \omega_{rf}\left(\frac{\Delta C}{C_s} - \frac{\Delta v}{v_s}\right)$$

Phase variation:

With $\dfrac{\Delta C}{C_s} = \alpha_c \delta$ and $\dfrac{\Delta v}{v_s} = \dfrac{\delta}{\gamma_s^2}$, we have

$$\dot{\phi} = \omega_{rf} \eta \delta$$

where $\eta = \alpha_c - \dfrac{1}{\gamma_s^2}$ the phase slippage factor, and momentum compaction factor $\alpha_c = \dfrac{1}{C} \oint \dfrac{D(s)}{\rho(s)} ds \approx \dfrac{1}{\nu_x^2}$.

Here the dispersion function $D(s)$ is defined as $x(s) = D(s)\delta$.

Synchrotron oscillation:

$$\begin{cases} \dot{\delta} = eV_0 \dfrac{\omega_0}{2\pi \beta_s^2 E_s} (\sin\phi - \sin\phi_s) \\ \dot{\phi} = \omega_{rf} \eta \delta \end{cases} \Rightarrow \frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h \omega_0^2 e V_0 \eta}{2\pi \beta_s^2 E_s} (\sin\phi - \sin\phi_s)$$

For $|\phi - \phi_s| \ll 1, \quad \Rightarrow \dfrac{d^2}{dt^2}(\phi - \phi_s) - \left(\dfrac{h \omega_0^2 e V_0 \eta}{2\pi \beta_s^2 E_s} \cos\phi_s \right) (\phi - \phi_s) \approx 0$

Stability condition:

$$\eta \cos\phi_s < 0$$

The **synchrotron frequency** : $\omega_s = \omega_0 \sqrt{-\dfrac{h e V_0 \eta}{2\pi \beta_s^2 E_s} \cos\phi_s}$, and **synchrotron tune:** $\nu_s = \dfrac{\omega_s}{\omega_0} = \sqrt{-\dfrac{h e V_0 \eta}{2\pi \beta_s^2 E_s} \cos\phi_s}$

Transition energy:

The longitudinal stability condition: $\eta \cos \phi_s < 0$ and $\eta = \alpha_c - \frac{1}{\gamma_s^2}$.

The transition energy: $\gamma_t = \frac{1}{\sqrt{\alpha_c}}$,

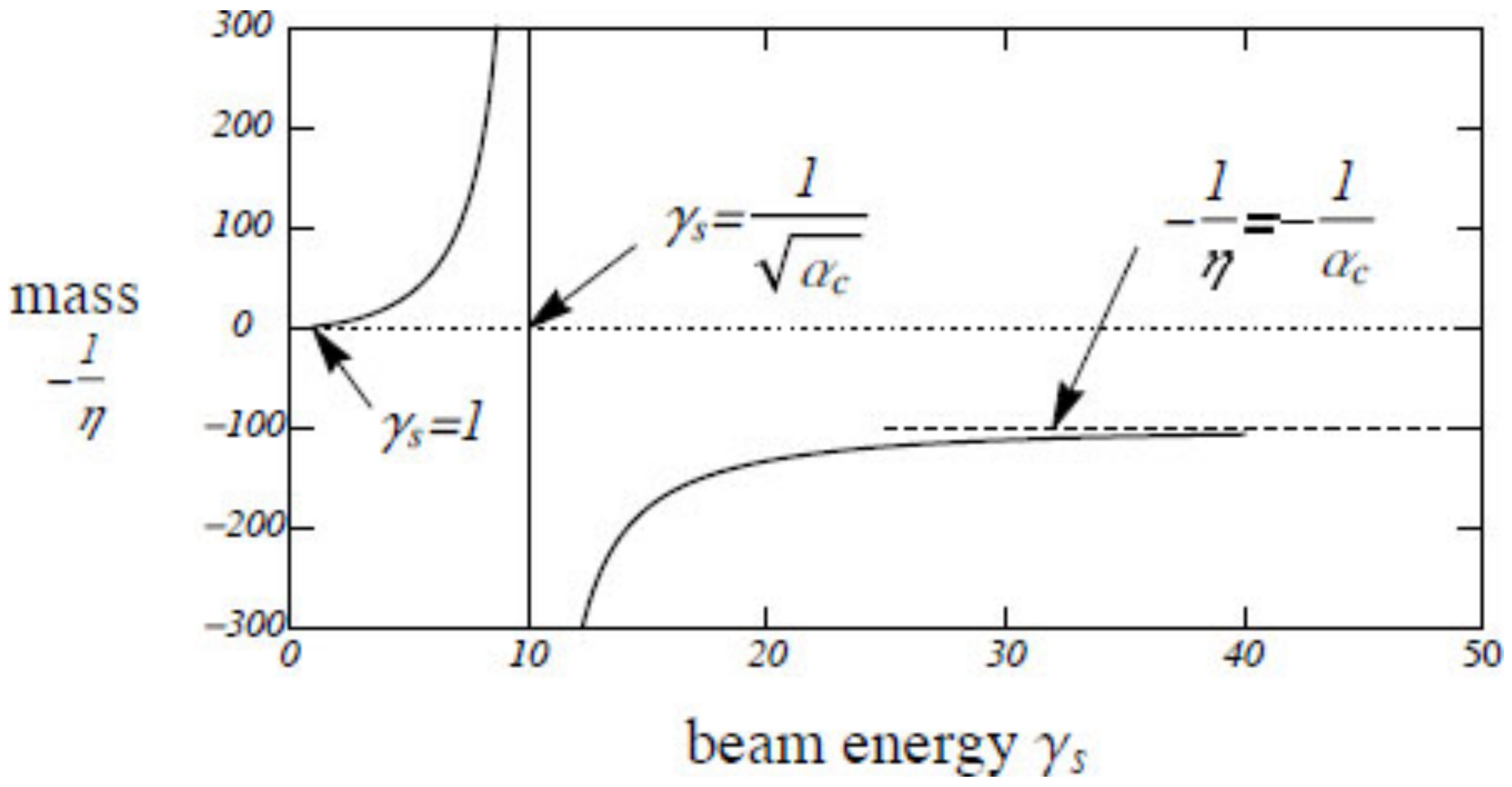
When $\gamma = \gamma_t, \eta = 0$.

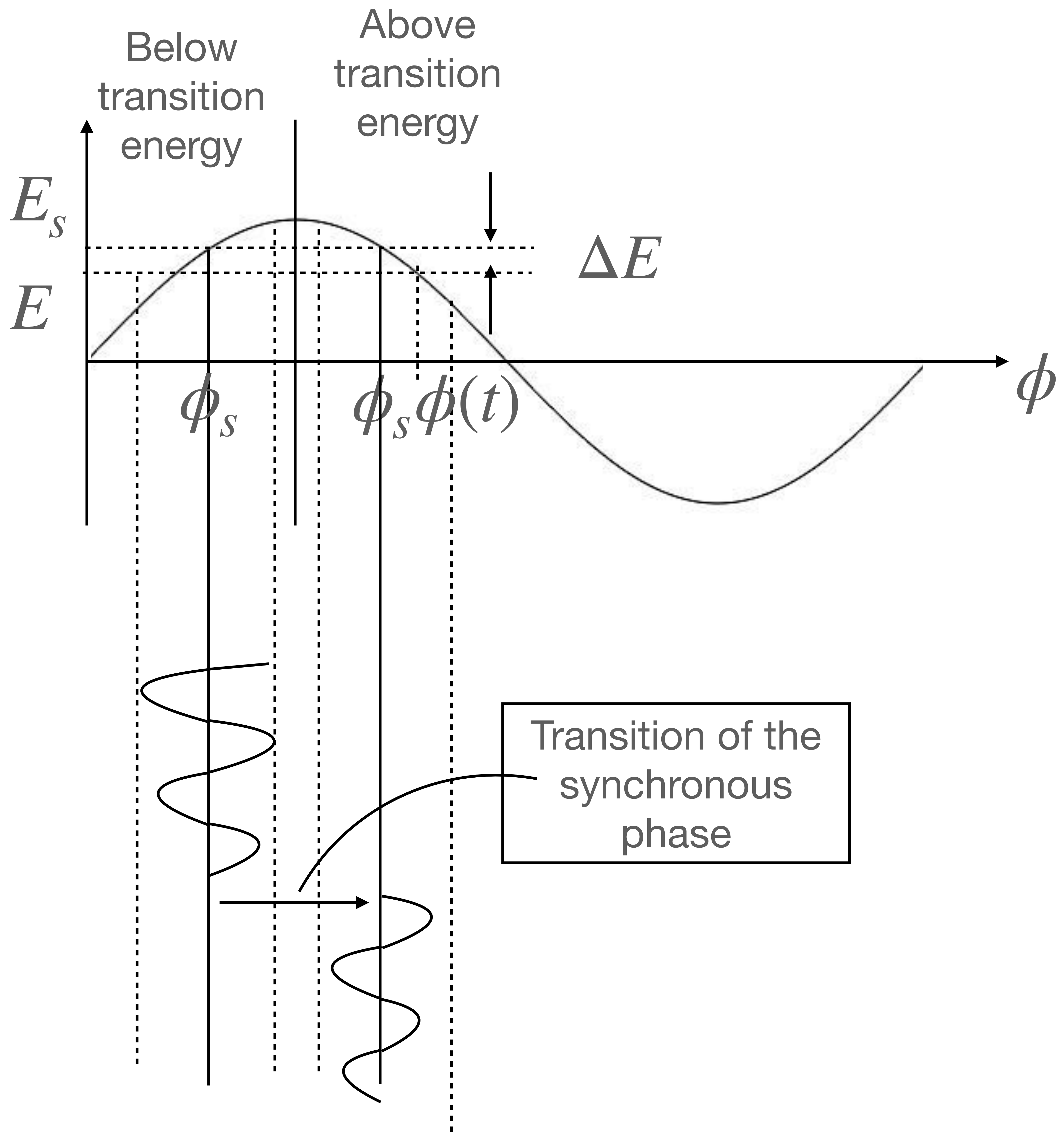
When $\gamma > \gamma_t, \eta > 0$, the stable phase at $\cos \phi_s < 0$.

When $\gamma < \gamma_t, \eta < 0$, the stable phase at $\cos \phi_s > 0$.

The phase stability also requires to choose $\phi_s = 0$ for $\eta < 0, \phi_s = \pi$ for $\eta > 0$

For electron synchrotron, the transition energy is about 5MeV, and for proton about 9 GeV.





RF buchet:

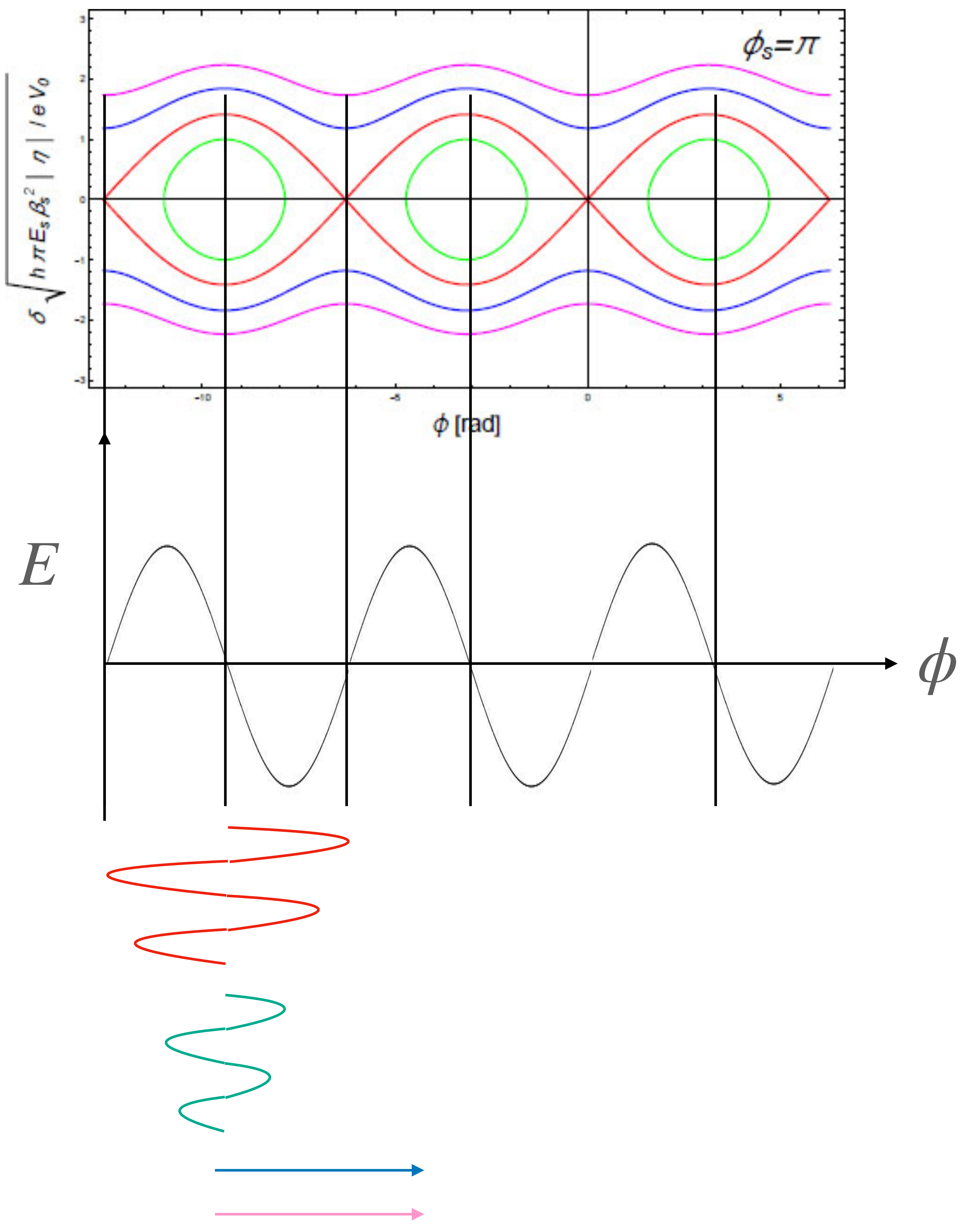
The Hamiltonian to describe the longitudinal motion:

$$H(\phi, \delta, t) = \frac{1}{2} h\omega_0 \eta \delta^2 + \frac{eV_0\omega_0}{2\pi\beta_s^2 E_s}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s].$$

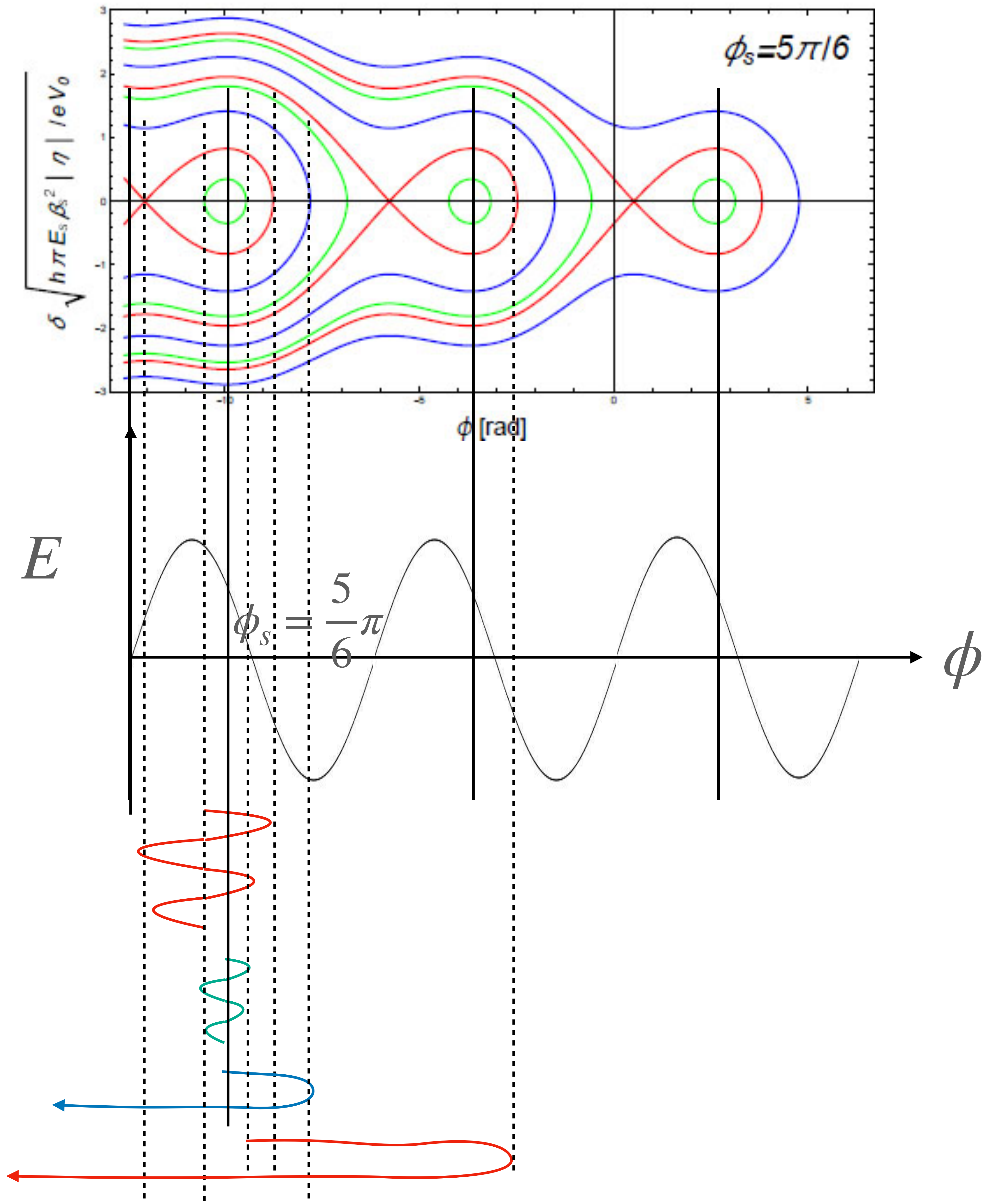
The Hamilton equation gives:

$$\begin{cases} \dot{\phi} = \frac{\partial H}{\partial \delta} = h\omega_0 \eta \delta \\ \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{eV_0\omega_0}{2\pi\beta_s^2 E_s}(\sin\phi - \sin\phi_s) \end{cases}.$$

For each $H(\phi, \delta)$, we can draw a curve on the (ϕ, δ) plane.



$\phi_s=\pi$
$\delta\sqrt{h\pi E_s\beta_s^2|\eta|/eV_0}$
ϕ [rad]
E
ϕ



$\phi_s=5\pi/6$
$\delta\sqrt{h\pi E_s\beta_s^2|\eta|/eV_0}$
ϕ [rad]
E
$\phi_s=\frac{5}{6}\pi$
ϕ

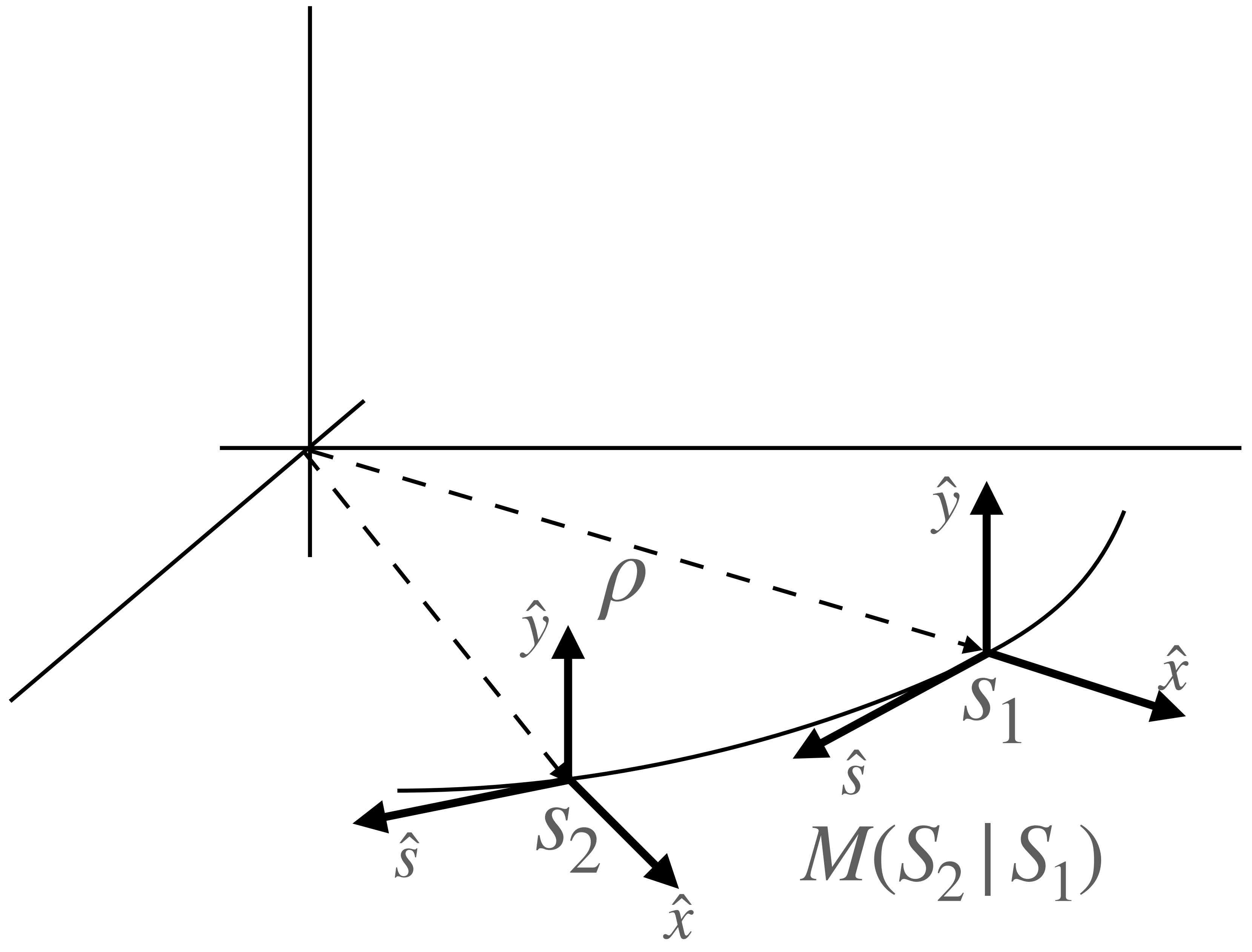
2.2 Transverse dynamics

Transverse dynamics is to study the particle movement at the xy plane. x is the radial direction, and y is the axial direction. Along the particle moving at the orbit, the xy plane is also changing.

With the definition of a 6-dimensional coordinate $(x, x', y, y', z, \delta)$, when the particle moving from position s_1 to s_2 , its coordinate changes to $(x_2, x_2', y_2, y_2', z_2, \delta_2)$.

$$\begin{bmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \\ z_2 \\ \delta_2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \end{bmatrix}, \text{ and } M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} & 0 & 0 \\ 0 & 0 & m_{43} & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & m_{56} \\ 0 & 0 & 0 & 0 & m_{65} & m_{66} \end{bmatrix}$$

is a 6×6 matrix, called **transfer matrix.** Here we assume that xyz is decoupled from each other. We then can deal with x,y,z separately now.



Transfer matrix: (x-direction)

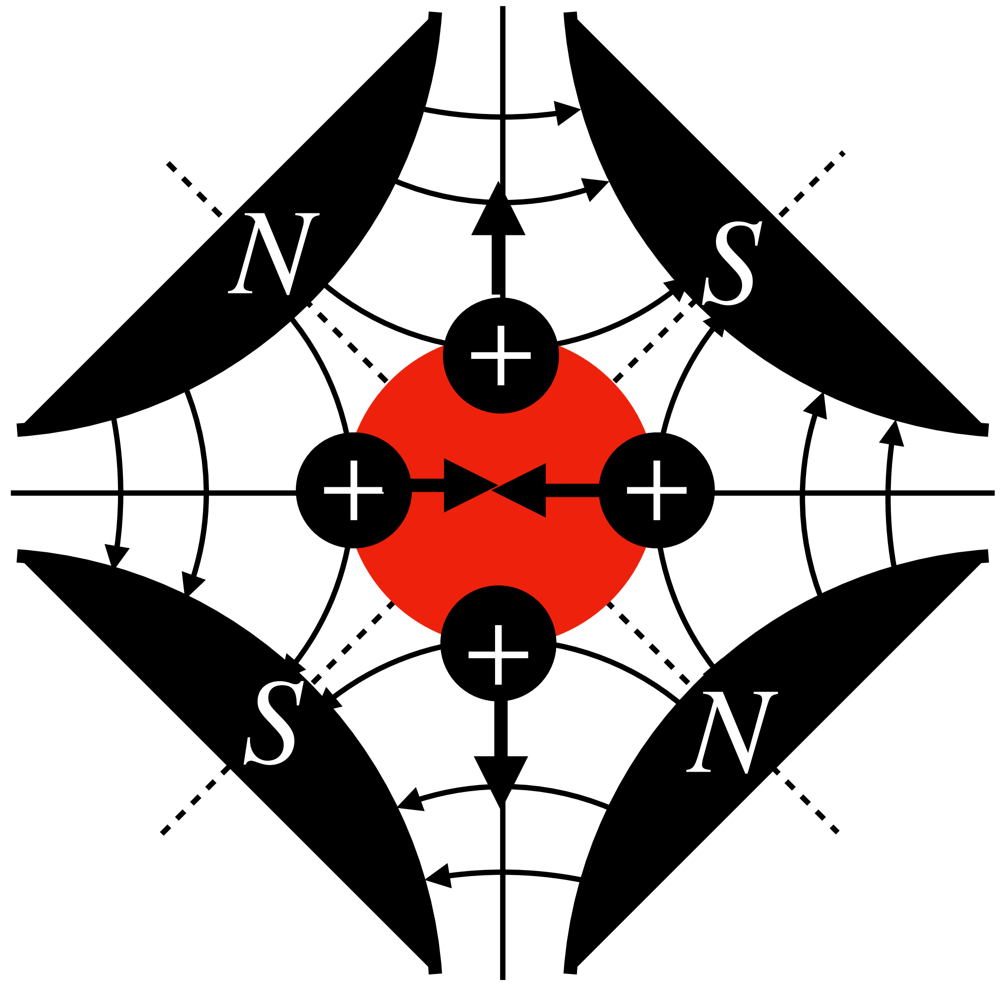
1) Thin-lens focusing and defocusing quadruples: $M_{qf} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} M_{qd} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$

2) Thick-lens focusing quadruple: $M_{qf} = \begin{bmatrix} \cos\sqrt{K}s & \frac{\sin\sqrt{K}s}{\sqrt{K}} \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s \end{bmatrix}, K = \frac{G}{B_0\rho}, \frac{e}{P} = \frac{1}{B_0\rho}$

3) Thick-lens defocusing quadruple: $M_{qd} = \begin{bmatrix} \cosh\sqrt{K}s & \frac{\sinh\sqrt{K}s}{\sqrt{K}} \\ -\sqrt{K}\sinh\sqrt{K}s & \cosh\sqrt{K}s \end{bmatrix}$

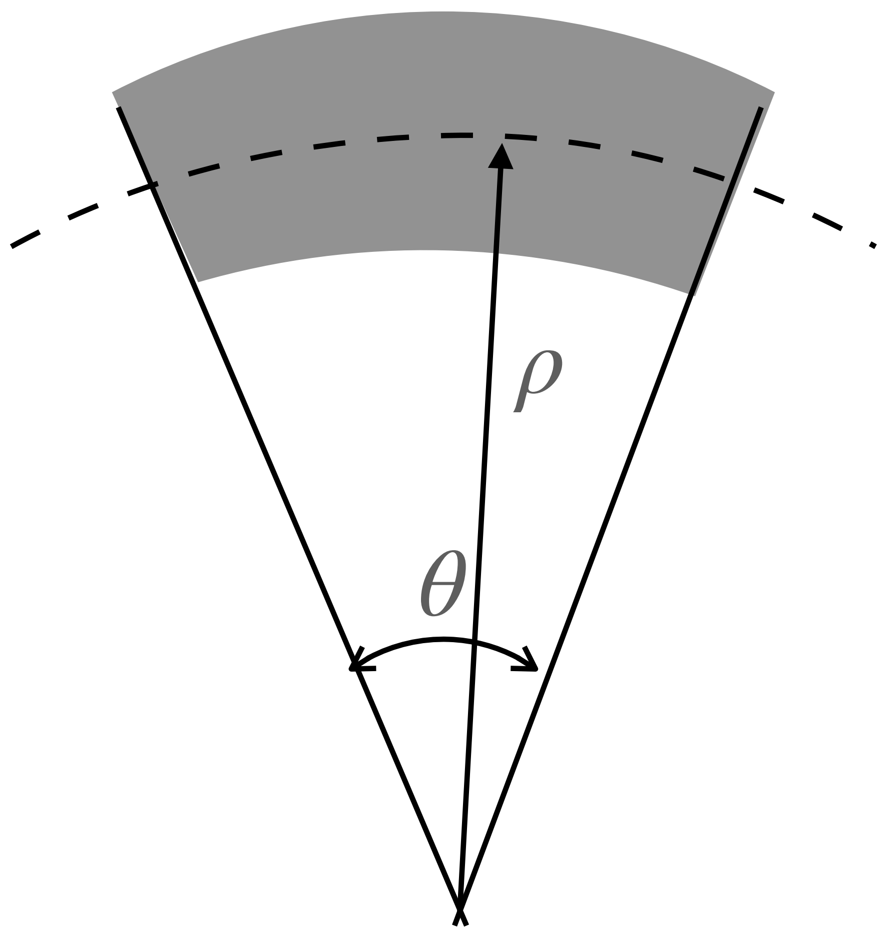
4) Drift of length d: $M_{drift} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$,

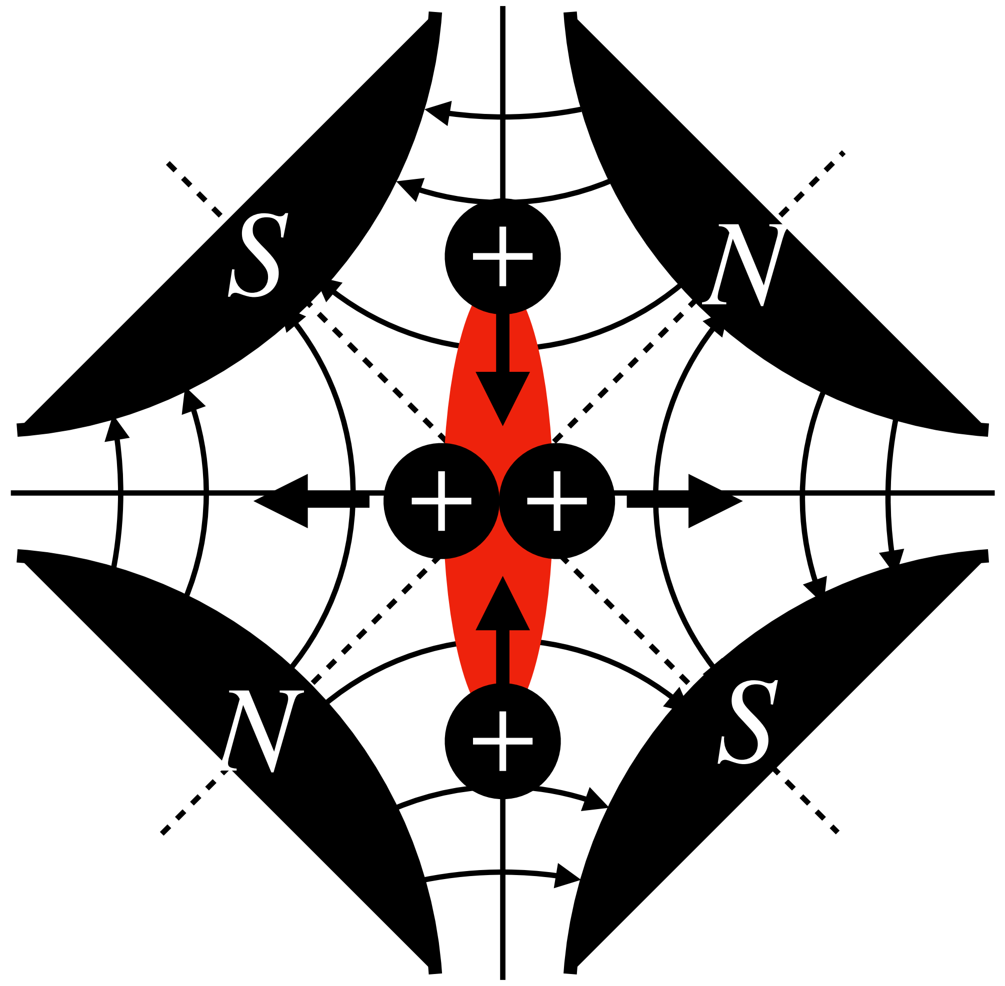
5) Sector dipole: $M_x(s\,|\,0) = \begin{bmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{bmatrix}, \theta = \frac{l}{\rho}$



$B_x = Gy$

$B_y = Gx$





$B_x = -Gy$

$B_y = -Gx$

Transfer matrix of a beam line:

The transfer matrix from s1 to s2 :

$M(S_2 | S_1) = M_m \cdots M_3 M_2 M_1$

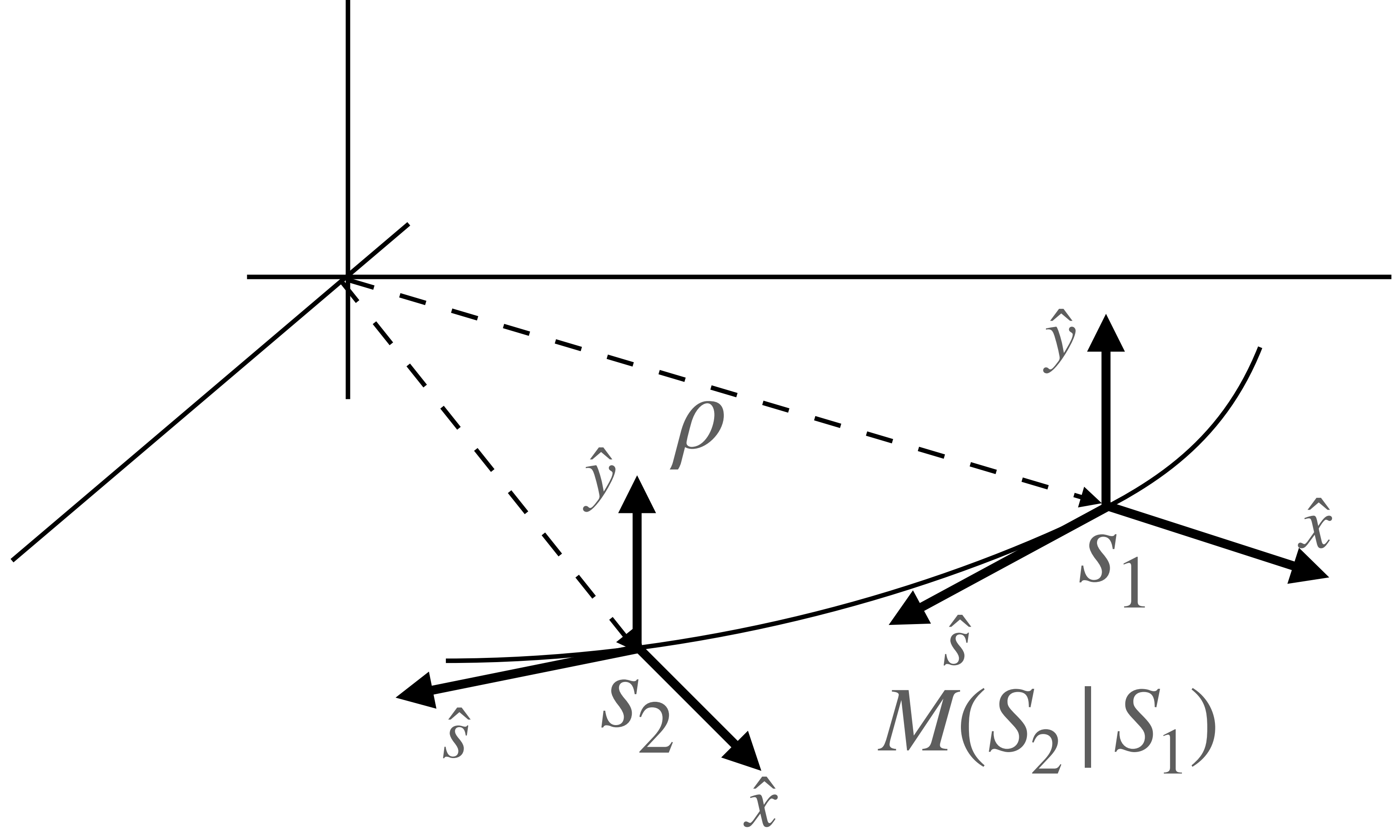
A one-turn map: $M(s + C | s)$

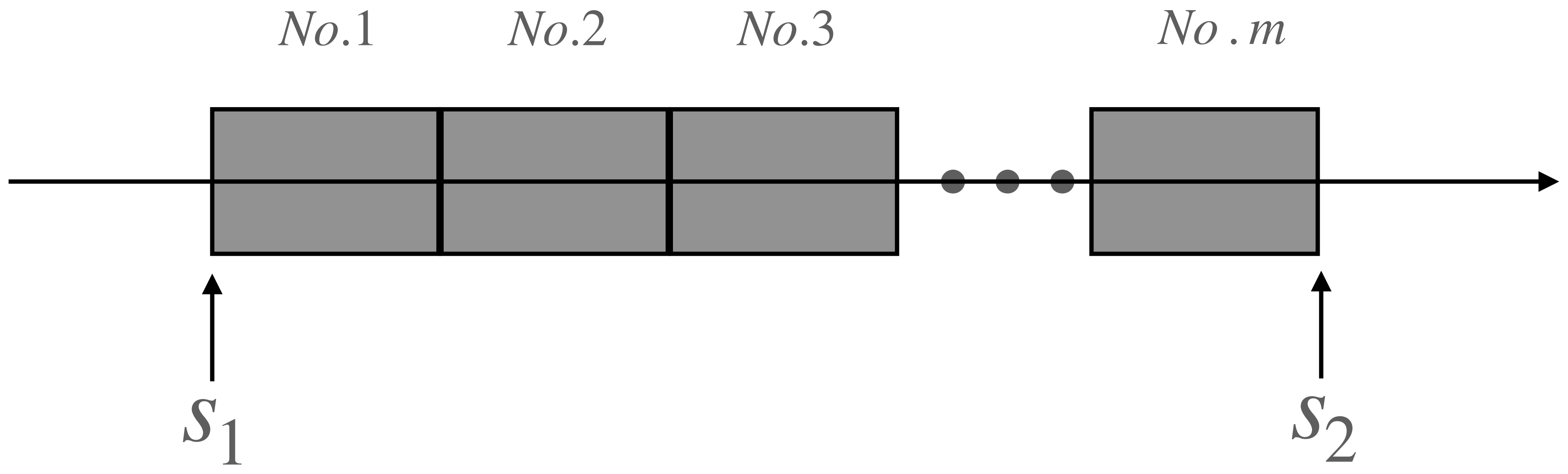
For a ring consists of p identical sections of length L, the one-turn map can be written as

$M(s + C | s) = M^p(s + L | s)$.

If the particle goes one turn and another position S' downstream of the ring, the map will be

$M(S' | S) M(S + C | S)$.

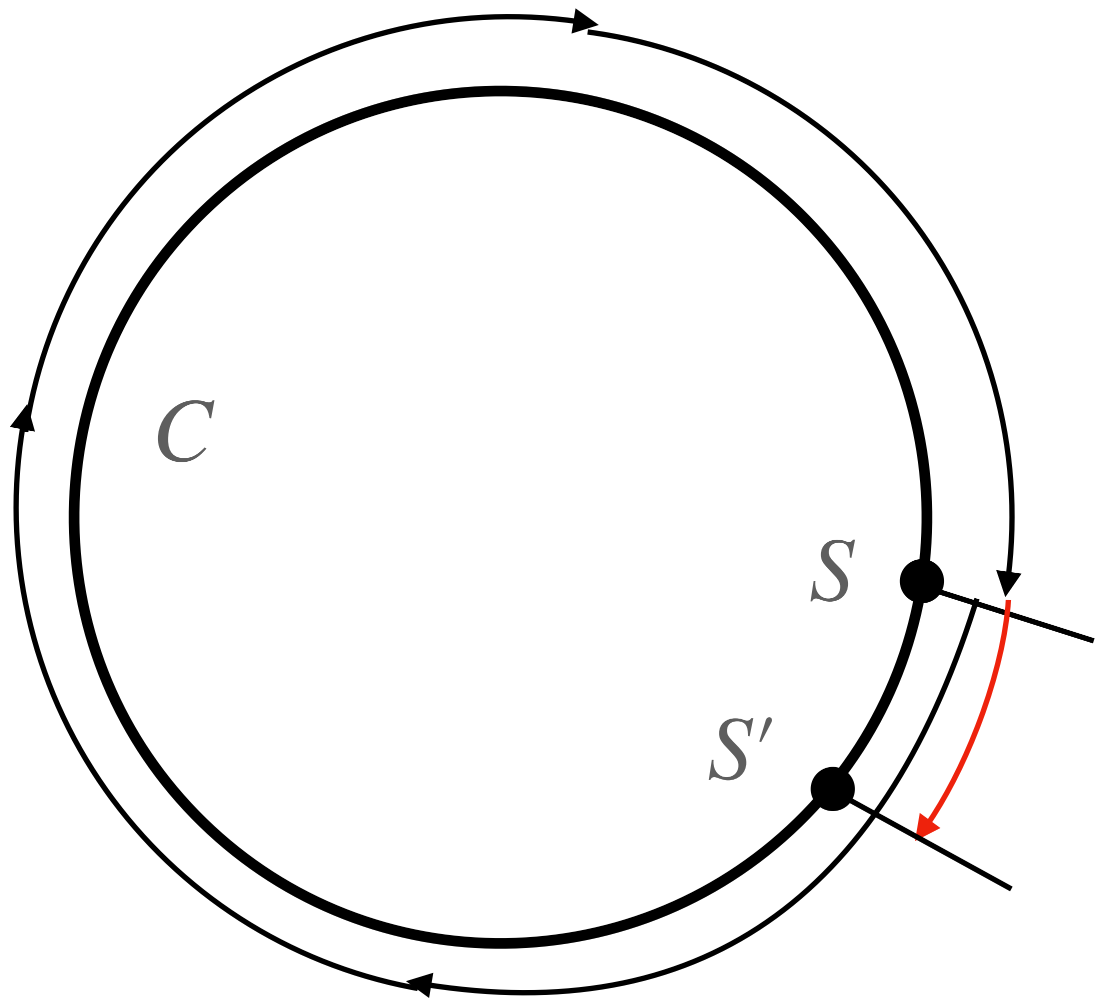




Transverse stability:

Stability of particle motion in a ring, means that when a particle with initial condition (x_0, x_0') is mapped repeatedly by one-turn map $M(s + C | s)$ for m turns, the resulting particle coordinates x and x' remains confined as $m \to \infty$.

Stability $\iff$ $M^m \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$ = finite as $m \to \infty$



Properties of the one-turn matrix:

(1)$M(s+C|s) = M(s|s-C)$

(2)$\det M(s+C|s) = 1$

(3)Tr $M(s+C|s)$ is constant with s

The one-turn matrix can be written as

$$M(s+C|s) = AI + BJ(s), \text{ with}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J(s) = \begin{bmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{bmatrix}.$$

$$A^2 + B^2[-\alpha^2(s) + \beta(s)\gamma(s)] = 1.$$

Let

$$-\alpha^2(s) + \beta(s)\gamma(s) = 1$$

Got

$$A^2 + B^2 = 1.$$

Let

$$A = \cos\mu, B = \sin\mu.$$

$$M(s+C|s) = I\cos\mu + J\sin\mu = \begin{bmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{bmatrix}.$$

The α, β, γ are the Twiss parameters, or Courant-Snyder parameters. And

$$J^2(s) = \begin{bmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{bmatrix} \begin{bmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Transverse Stability Condition:

It can be proved that

$$M(s+C\,|\,s) = I\cos\mu + J\sin\mu = \exp[J(s)\mu]$$

After m turns, the transfer matrix will be

$$M^m(s+C\,|\,s) = \exp[J(s)m\mu] = I\cos(m\mu) + J(s)\sin(m\mu).$$

Only if μ is real, $M^m(s+C\,|\,s)$ remains confined as $m \to \infty$.

The stable condition is:

$$|\,\mathrm{Tr}\,M(s+C\,|\,s)\,| = M_{11} + M_{22} = |\,2\cos\mu\,| \leq 2$$

Definition: *The matrix exponential function*

If $\mathbf{a}$ is an $m \times m$ matrix, then $\exp\mathbf{a}$ is the $m \times m$ matrix defined by

$$\exp\mathbf{a} = \mathbf{1} + \sum_{j=1}^{\infty} \mathbf{a}^j / j!$$

Symplecticity

For a **linear** 1-D motion (2-D phase space) , the symplecticity condition is the determinant of the 2x2 transfer matrix M must be equal to 1.

For the n-D case, a linear motion can be described by a $2n \times 2n$ matrix. And the symplecticity condition:

$$\tilde{M}SM = S,$$

where

$$S = \begin{bmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & S_0 \end{bmatrix}, \text{ and } S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Eigenvalues of matrix M

Let $v_{1,2}$ be the eigenvectors of M with eigenvalues $\lambda_{1,2}$.

$Mv_1 = \lambda_1 v_1$, $Mv_2 = \lambda_2 v_2$.

Construct two matrices:

$V = [v_1 \quad v_2]$, and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, so

$$MV = V\Lambda$$

$$M = V\Lambda V^{-1}$$

and

$$M^m = V\Lambda^m V^{-1} = V \begin{bmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{bmatrix} V^{-1}$$

Stability needs: $|\lambda_1| \leq 1$ **, and** $|\lambda_2| \leq 1$**,**

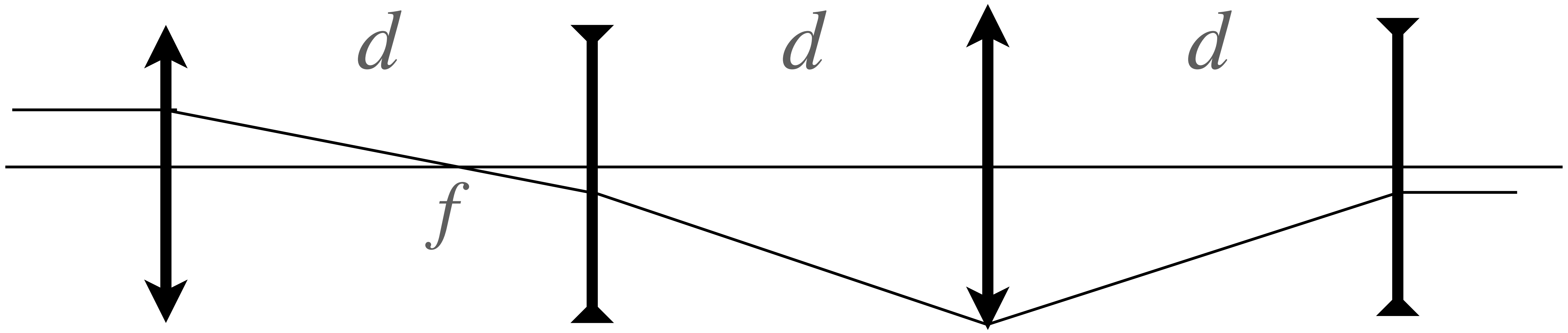
For a **linear** 1-D system,

$$\det(M - \lambda I) = \begin{vmatrix} m_{11} - \lambda & m_{22} \\ m_{21} & m_{22} - \lambda \end{vmatrix} = \lambda^2 - (\mathrm{Tr}M)\lambda + 1 = 0$$

where $\mathrm{Tr}M = m_{11} + m_{22}$, and $|\lambda| \leq 1$ for its two solutions if and only if $|\mathrm{Tr}\, M| \leq 2$

An example: Periodical FODO structure of thin-lens approximation

$$M_{qf} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \quad M_{qd} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$$



$$M_{drift} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$M = M_{drift} M_{qd} M_{drift} M_{qf} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f} - \frac{d^2}{f^2} & 2d + \frac{d^2}{f} \\ -\frac{d}{f^2} & 1 + \frac{d}{f} \end{bmatrix}$$

$$\mathrm{Tr} M = m_{11} + m_{22} = 2 - \frac{d^2}{f^2}$$

$$2 - \frac{d^2}{f^2} \geq -2 \quad \Rightarrow \quad \frac{d}{f} \leq 2$$

$$2 - \frac{d^2}{f^2} \leq 2 \quad \Rightarrow$$ Satisfied automatically

Homeworks:

1. Prove the transverse weak focusing condition, $0 < n < 1$, for the magnetic fields $B_z(r) = B_z(r_s)\frac{1}{r^n}$.
2. Derive the transfer matrix in a thick focusing quadrupole.
3. Describe the working principle of a Rhodotron, and try to give its synchronous condition.