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From Classical Electrondynamics to Advanced Accelerator Light Sources

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Outline

Maxwell's Equations

- Differential form of Maxwell's equations

Gauss's law for electricity: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss's law for magnetism: $$\nabla \cdot \mathbf{B} = 0$$

Faraday's law of electromagnetic induction: $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere - Maxwell law: $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Scalar and vector potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \ \mathbf{B} = \nabla \times \mathbf{A}$$

- Gauge transformation keep the electric and magnetic fields invariant

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \ \phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

Solution of Maxwell's Equations

- Taking Lorenz gauge $\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$, Maxwell's equations can be cast into a non-homogeneous wave equation

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

  with $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ the speed of light in vacuum.
- Retarded Green's function of wave equation

$$G_{\text{ret}}(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{4\pi R}\delta\left(t - t' - \frac{R}{c}\right), \text{ with } R \equiv |\mathbf{r} - \mathbf{r}'|$$

  This represents a spherical wave propagating outward from the source point $\mathbf{r}'$ at time t', arriving at $\mathbf{r}$ at time $t = t' + R/c$.
- The retarded potentials of a charge and current distribution are

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_{\text{ret}})}{R} d^3 r' + \phi_0$$
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_{\text{ret}})}{R} d^3 r' + \mathbf{A}_0$$

  with the retarded time given by $t_{\text{ret}} = t - \frac{R}{c}$.

Solution of Maxwell's Equations

- For of a point charge one may expect $\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{e}{R}$, but this is wrong.
- Reason: for each volume element V_i of the charge distribution, we are to take ρ at the retarded time $t_{i,\mathrm{ret}} = t_i - R_i/c$, but since the charge is moving, it is in a different place for each volume element V_i!

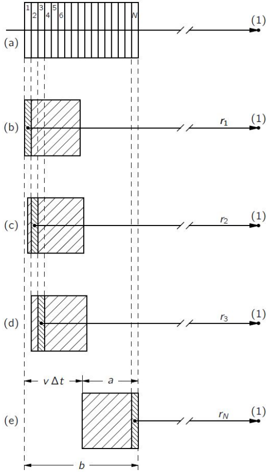


Fig. 21-6. Integrating $\rho(t - r'/c)\,dV$ for a moving charge.

Liénard-Wiechart Potentials

- The correct result is (Liénard-Wiechart potentials)

$$\phi(\mathbf{r},t)=\frac{e}{4\pi\epsilon_0}\frac{1}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}\bigg|_{\text{ret}},\quad \mathbf{A}(\mathbf{r},t)=\frac{e}{4\pi\epsilon_0}\frac{1}{c}\frac{\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}\bigg|_{\text{ret}}$$

  with the retarded time given by $t_{\text{ret}}=t-\frac{R}{c}$.

- The electromagnetic field is given by

$$\mathbf{E}(\mathbf{r},t)=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}=\frac{e}{4\pi\epsilon_0}\left\{\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2R^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}+\frac{\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{cR(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\right\}\bigg|_{\text{ret}}$$

$$c\mathbf{B}(\mathbf{r},t)=c\left(\nabla\times\mathbf{A}\right)=\hat{\mathbf{n}}\times\mathbf{E}\big|_{\text{ret}}$$

  The second term of the electric field is usually referred to as the radiation field based its $\mathcal{O}\left(\frac{1}{R}\right)$ dependence.

- The Doppler factor $(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})$ makes the radiation field in the forward direction stronger than that in the backward direction. When $\beta\to 1(\gamma\gg 1)$, the radiation will be dominantly in the forward direction with an opening angle of $\frac{1}{\gamma}$.

Heaviside-Feynman Decomposition

- We can also express the above result in another equivalent elegant form (Feynman Lecture, Vol II, Eq. (21.1))

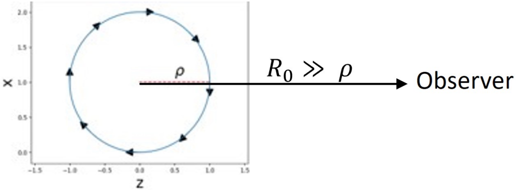
$$\mathbf{E}(\mathbf{r},t) = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{e}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}(t_{\rm ret})}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}(t_{\rm ret})}{R^2}\right) + \frac{1}{c^2}\frac{d^2\hat{\mathbf{n}}(t_{\rm ret})}{dt^2}\right]$$

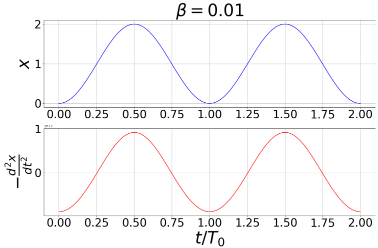
  The first twos terms are of $\mathcal{O}\left(\frac{1}{R^2}\right)$, the third term can be viewed as as radiation.
- Take far-field approximation, the radiation field can be expressed as

$$\mathbf{E}(\mathbf{r},t) \approx -\frac{e}{4\pi\epsilon_0 R_0}\frac{1}{c^2}\left(\frac{d^2x(t_{\rm ret})}{dt^2}\hat{x} + \frac{d^2y(t_{\rm ret})}{dt^2}\hat{y}\right)$$

  Interpretation: it is the apparent acceleration of particle motion perpendicular to the observation direction that determines the radiation field.

Application of Heaviside-Feynman Formula to Bending Magnet Radiation



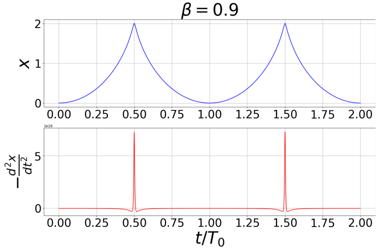


$$E_x \propto \frac{d^2 x(t_{\text{ret}})}{dt^2}$$

$$x(t_{\text{ret}}) = \rho\left[1 - \cos\left(\frac{\beta c t_{\text{ret}}}{\rho}\right)\right]$$

$$t \approx t_{\text{ret}} + \frac{R_0 - z(t_{\text{ret}})}{c}$$

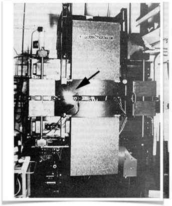
$$= \frac{R_0}{c} + t_{\text{ret}} + \frac{\rho}{c}\sin\left(\frac{\beta c t_{\text{ret}}}{\rho}\right)$$

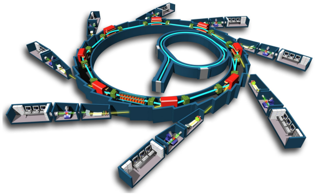


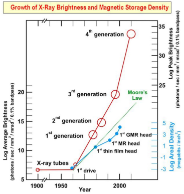
- When $\beta \ll 1$ the radiation frequency is close to the particle revolution frequency.
- When β approaches 1 (relativistic), the radiation pulse become ultrashort, whose frequency can extend to that much higher than the revolution frequency!

Discovery of Synchrotron Radiation and Birth of Accelerator Light Sources

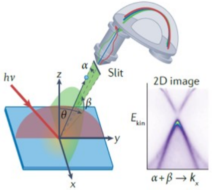
- The theoretical prediction was confirmed at GE's 70 MeV synchrotron accelerator in 1947, and this type of radiation is thus named as synchrotron radiation. It turns out to be a very effective way to generate high-brightness X-rays.







- Accelerator-based light source is a powerful tool. A good example is the today-announced 2025 Future Science Prize: for prediction and realization of topological electronic materials. In particular, the Weyl Fermion is experimentally discovered at Shanghai Synchrotron Radiation Facility (SSRF) using ARPES.



Radiation Characteristics from a Point Charge and a Particle Beam

- Single particle radiation

$$\frac{d^2W}{d\omega d\Omega} = \frac{e^2\omega^2}{16\pi^3\epsilon_0 c^3}\left| \int_{-\infty}^{\infty} [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{v})]\, e^{i\omega(t' - \hat{\mathbf{n}}\cdot\vec{r}(t')/c)} dt' \right|^2 .$$

- What we learn: once the prescribed motion of electron is given, regardless of the origin of this motion, either electric force, magnetic force, or gravitational force (as long as the curvature of the spacetime itself is not significant), the radiation property is determined. In classical electrodynamics, radiation can be viewed as a geometric effect.

- For a bunch with N electrons, the superimposed radiation is

$$\frac{d^2W}{d\omega d\Omega} = \frac{e^2\omega^2}{16\pi^3\epsilon_0 c^3}\left| \int_{-\infty}^{\infty} e^{i\omega t'} \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \left(\sum_{j=1}^{N} \vec{v}_j(t') e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\vec{r}_j(t')} \right) \right] dt' \right|^2 .$$

- What we learn: the radiation from a particle beam is determined by both the reference electron prescribed motion and the beam phase space distribution.

Incoherent Radiation and Coherent Radiation

- For beam with a small angular divergence, the impact of the phase terms for difference particles are much more significant that their velocity terms. Then

$$\left.\frac{d^2W}{d\omega d\Omega}\right|_{\text{beam}} = \left.\frac{d^2W}{d\omega d\Omega}\right|_{\text{point}} N_e^2 |b(\omega)|^2,$$

$$\text{Bunching factor}: \quad b(\omega) = \frac{1}{N_e}\sum_{j=1}^{N_e} e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\vec{r}_j} = \frac{1}{N_e}\sum_{j=1}^{N_e} e^{-i\frac{\omega}{c}\left(x_j \sin\theta\cos\varphi + y_j \sin\theta\sin\varphi + z_j\right)}.$$

- For relativistic particles, the radiation is dominantly in the forward direction, we can further simplify the bunching factor as the Fourier transform of longitudinal charge density distribution

$$b(\omega) = \frac{1}{N_e}\sum_{j=1}^{N_e} e^{-i\frac{\omega}{c}z_j} = \int_{-\infty}^{\infty} \rho(z) e^{-i\frac{\omega}{c}z} ds.$$

- Incoherent radiation: phase randomly distributed, $\langle b(\omega)\rangle = \frac{1}{N_e}$
- Coherent radiation: radiation from different electrons add in phase, $\langle b(\omega)\rangle \sim 1$

Normal bunch

Incoherent radiation $P \propto N_e$

Microbunching

Coherent radiation $P \propto N_e^2$

Main Pursuit of Accelerator Light Sources: higher brightness, better coherence, at ever shorter wavelength

- Brightness: density of photons in 6D phase space

$$\mathcal{B} = \frac{dN_{\rm ph}/dt}{4\pi\Sigma_x\Sigma_{x'}\Sigma_y\Sigma_{y'}0.1\%\Delta\omega/\omega}$$

- Coherence: phase relation between difference spatial and temporal parts of the light. Usually divided into transverse (spatial) and longitudinal (temporal) coherence
  - Diffraction limited: $\Sigma_x = \sqrt{\sigma_{x,e}^2 + \sigma_{x,p}^2}$, $\Sigma_{x'} = \sqrt{\sigma_{x',e}^2 + \sigma_{x',p}^2}$, $\sigma_{x,e} < \sigma_{x,p}$, $\sigma_{x',e} < \sigma_{x',p}$, $\epsilon_{\perp,e} < \frac{\lambda_R}{4\pi}$.
  - Fourier transform limited: $\Delta t \Delta\omega \geq \frac{1}{2}$.
- Strategies to realize higher brightness:
  - lower the transverse emittance of the electron beam (low-emittance ring)
  - make the radiation become more narrow-banded (shape the electron trajectory, for example undulator as the radiator, or shape the beam using laser)
  - increase the number of photons radiated (shape the electron beam longitudinally, microbunching, free-electron lasers)
- Other pursuits: ultrashort pulse, light with orbit angular momentum, etc

Research Directions of Accelerator Light Sources

- Direction 1: Tailor the Electron Trajectory
  - Bending Magnet as Radiator: Synchrotron Radiation
  - Undulator Magnet as Radiator: Undulator Radiation
  - Laser as Radiator: Inverse Compton Scattering
- Direction 2: Tailor the Electron Beam Distribution
  - Longitudinal Microbunching & Coherent Radiation Generation
  - Helical Microbunching & OAM Light Generation
  - Beam Conditioning
- Direction 3: Radiation Acting Backing on Electron Beam
  - Low-Emittance Storage Rings: balance of quantum excitation and radiation damping
  - Low-Longitudinal-Emittance Storage Rings & SSMB
  - Free-Electron Lasers

Bending Magnet Radiation

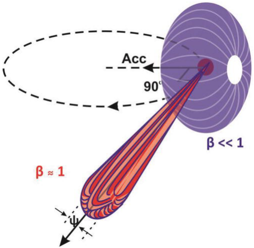
- The spectral-spatial distribution of the synchrotron (bending magnet) radiation

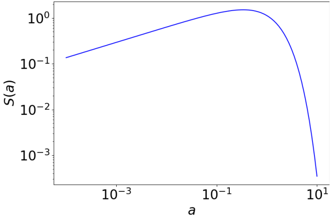
$$\frac{d^2W}{d\omega d\Omega} = \frac{3e^2}{16\pi^3\epsilon_0 c}\gamma^2\left(\frac{\omega}{\omega_c}\right)^2\left(1+\gamma^2\theta^2\right)^2\left[K^2_{2/3}(\xi)+\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}K^2_{1/3}(\xi)\right],$$

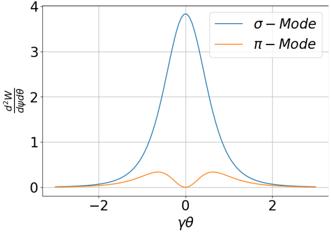
  with the characteristic frequency $\omega_c = \frac{3}{2}\frac{\gamma^3 c}{\rho}$, $\xi = \frac{\omega}{2\omega_c}\left(1+\gamma^2\theta^2\right)^{\frac{3}{2}}$.
- The overall radiation spectrum

$$\frac{dW}{d\psi d\omega} = \frac{e^2\gamma}{9\pi\epsilon_0 c}S\left(\frac{\omega}{\omega_c}\right)$$

  with $S(a) = \frac{9\sqrt{3}}{8\pi}a\int_a^\infty dx' K_{\frac{5}{3}}(x')$ being a universal function.







- Opening angle $\frac{1}{\gamma}$. Broadband. σ-mode (horizontal polarization) dominates.

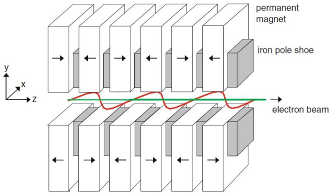
Undulator Radiation

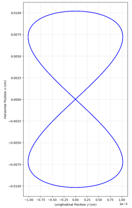
- Prescribed motion of electron in a planar undulator

$$B_y = B_0 \sin(k_u s),\ \ s = \bar{v}_z t',\ \ \bar{v}_z = \left(1 - \frac{1+K^2/2}{2\gamma^2}\right) c,$$

$$v_x = \frac{Kc}{\gamma}\cos(k_u s),\ \ v_z = \bar{v}_z - \frac{K^2 c}{4\gamma 2}\cos(2k_u s)$$

$$x = \frac{K}{\gamma k_u}\sin(k_u s),\ \ z = \bar{v}_z t - \frac{K^2 c}{8\gamma 2 k_u}\sin(2k_u s),$$





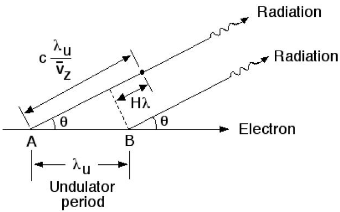
with $K = \frac{eB_0}{m_e c k_u}$ being the dimensionless undulator parameter.

- Resonance condition: the radiation overtakes the electron for one or multiple wavelength when the electron traverse in the undulator for one period

$$c\frac{\lambda_u}{\bar{v}_z} - \lambda_u \cos\theta = H\lambda_R$$

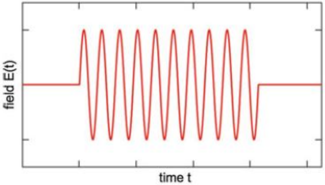
$$\Rightarrow \lambda_R = \frac{1 + K^2/2 + \gamma^2\theta^2}{2\gamma^2 H}\lambda_u,$$

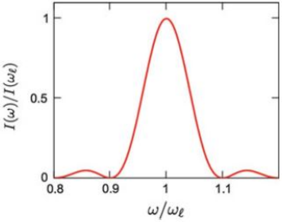
H is the harmonic number



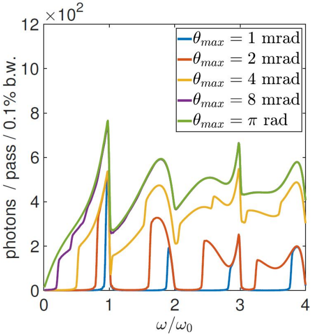
Undulator Radiation Spectrum - Qualitative

- On-axis radiation bandwidth $\frac{\Delta\omega}{\omega} = \frac{1}{HN_u}$ for H-th harmonic. Opening angle $\frac{1}{\sqrt{N_u}\gamma}$





- Typical overall radiation spectrum, including the off-axis redshifted part



Undulator Radiation Spectrum - Quantitative

- Planar undulator radiation

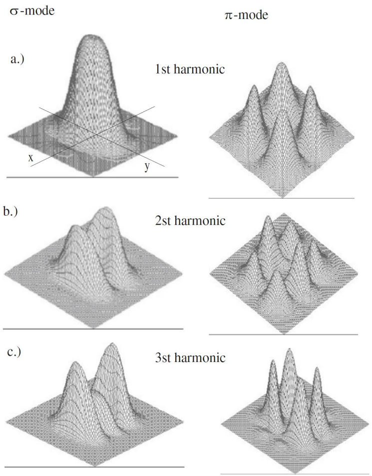
$$\frac{d^2W}{d\omega d\Omega} = \frac{e^2\omega^2 L_u^2}{16\pi^3\epsilon_0 c^3}\left(\left|\sum_{H=1}^{\infty}\frac{\sin(N_u\pi\varepsilon_H)}{N_u\pi\varepsilon_H}\left(\frac{K}{\gamma}\mathcal{D}_{1H} + \sin 2\theta\cos\varphi\mathcal{D}_{2H}\right)\right|^2 + \left|\sum_{H=1}^{\infty}\frac{\sin(N_u\pi\varepsilon_H)}{N_u\pi\varepsilon_H}\sin 2\theta\sin\varphi\mathcal{D}_{2H}\right|^2\right),$$

$$\varepsilon_H \equiv \frac{\omega}{\omega_1} - H = \frac{\omega}{2ck_u\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\theta^2\right) - H,$$

$$\mathcal{D}_{1H} \equiv -\sum_{m=-\infty}^{\infty} J_{H-2m+1}(\zeta)\left[J_{m-1}(\eta) + J_m(\eta)\right],$$

$$\mathcal{D}_{2H} \equiv \sum_{m=-\infty}^{\infty} J_{H-2m}(\zeta)\, J_m(\eta),$$

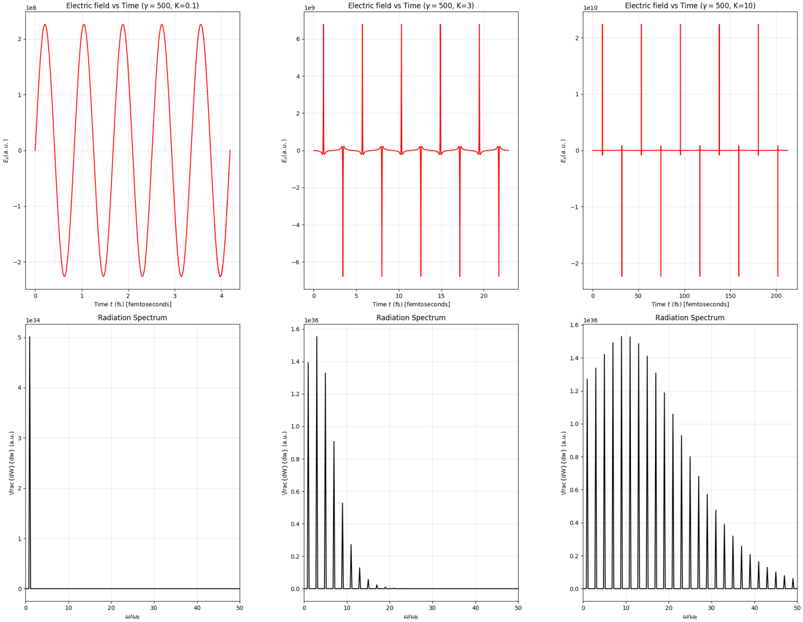
$$\zeta \equiv -\frac{\omega}{c}\sin\theta\cos\varphi\frac{K}{\gamma k_u},\quad \eta \equiv \frac{\omega}{c}\cos\theta\frac{K^2}{8\gamma^2 k_u}.$$



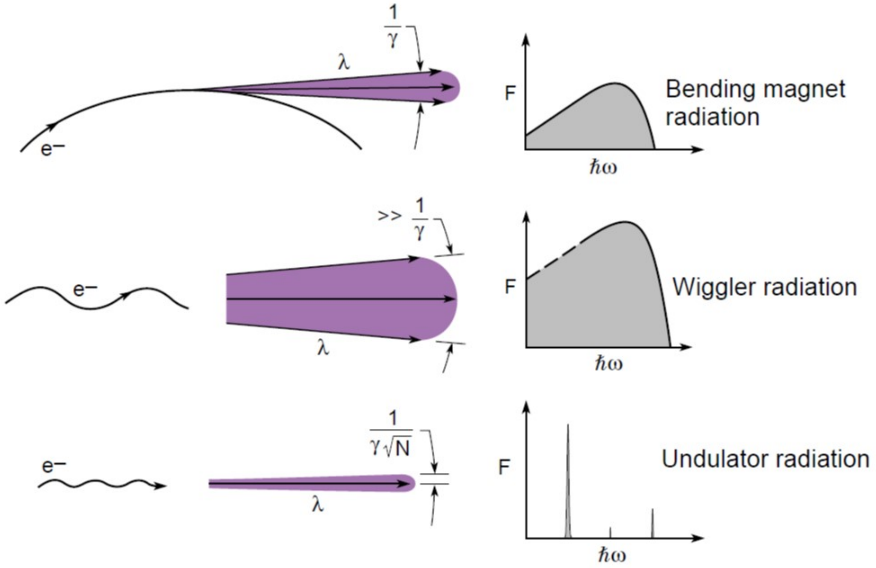
- σ-mode dominates.

From Undulator Radiation to Wiggler Radiation

- Undulator $K \lesssim 1$, wiggler $K \gtrsim 10$

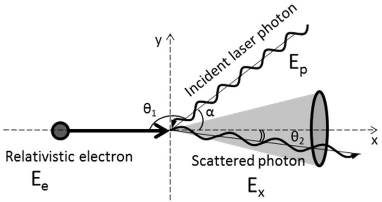


Qualitative Comparison



Optical Laser as the Radiator: Inverse Compton Scattering

- Analog with the undulator's sinusoidal magnetic field, we can use laser as the radiator. This is called inverse Compton scattering (ICS) radiation source



- For head-on collision, we have (note that the laser propagates at the speed of c)

$$\lambda_R = \frac{1 + a_0^2/2 + \gamma^2\theta^2}{4\gamma^2 H}\lambda_L$$

$a_0 = 0.855 \times 10^{-9}\lambda_L[\mu\text{m}]I_0^{\frac{1}{2}}[\text{W/cm}^2]$ a dimensionless laser strength parameter.

- Advantage: the radiation can extend to ever shorter wavelength such as γ-ray
- Linear Compton scattering: $a_0 \ll 1$, single photon absorption, similar to undulator radiation with $K \ll 1$
- Nonlinear Compton scattering: $a_0 \gtrsim 1$, multiple photon absorption, similar to wiggler radiation

Research Directions of Accelerator Light Sources

- Direction 1: Tailor the Radiator Field or Electron Trajectory
  - Bending Magnet as Radiator: Synchrotron Radiation
  - Undulator Magnet as Radiator: Undulator Radiation
  - Laser as Radiator: Inverse Compton Scattering
- Direction 2: Tailor the Electron Beam Distribution
  - Longitudinal Microbunching
  - Helical Microbunching & OAM Light Generation
  - Beam Conditioning
- Direction 3: Radiation Acting Backing on Electron Beam
  - Low-Transverse-Emittance Storage Rings & DLSR
  - Low-Longitudinal-Emittance Storage Rings & SSMB
  - Free-Electron Lasers

Longitudinal Microbunching for Coherent Radiation Generation

- Radiation of N electrons

$$\left.\frac{d^2W}{d\omega d\Omega}\right|_{\text{beam}} = \left.\frac{d^2W}{d\omega d\Omega}\right|_{\text{point}} N_e^2 |b(\omega)|^2,$$

$$b(\omega) = \frac{1}{N}\sum_{j=1}^{N} e^{-i\frac{\omega}{c}z_j} = \int_{-\infty}^{\infty} \rho(z) e^{-i\frac{\omega}{c}z} ds.$$

- Incoherent radiation: phase randomly distributed, $\langle b(\omega)\rangle = \frac{1}{N_e}$
- Coherent radiation: radiation from different electrons add in phase, $\langle b(\omega)\rangle \sim 1$
- Microbunching: bunch length or substructure's length smaller than the radiation wavelength. Mathematically, it means the Fourier transform of the current distribution should have non-vanishing value at the radiation wavelength

Laser is the Most Useful Tool for Electron Phase Space Manipulation

- Laser-electron interaction in an undulator:

$$\vec{E} = -\hat{x}E_0 \sin(\omega_L t - k_L z + \psi_0)$$
$$\frac{d\mathcal{W}}{dt} = e\vec{v}\cdot\vec{E} = \frac{KceE_0}{\gamma}\sin k_u z \sin(\omega_L t - k_L z + \psi_0)$$
$$= -\frac{KceE_0}{2\gamma}[\cos(k_u z + \omega_L t - k_L z + \psi_0) - \cos(k_u z - \omega_L t + k_L z - \psi_0)]$$

- In order for the first cosine term to deliver or extract energy systematically from the electron, we need $k_u\bar{v}_z + kc - k\bar{v}_z = 0$ or $\frac{\bar{v}_z}{c} = \frac{k}{k-k_u}$ which is impossible.
- The second term is nonoscillatory only when

$$\boxed{\frac{\bar{v}_z}{c} = \frac{k}{k+k_u} \Rightarrow \lambda_L = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right).}$$

which is exactly the resonance condition for on-axis radiation.

- This is an extremely important result: the condition for sustained energy transfer all along the undulator yields the exactly same light wavelength as is observed in undulator radiation at $\theta = 0$. This fact is the reason why spontaneous undulator radiation can serve as seed radiation in an FEL as will be introduced soon.

Building Blocks of Phase Space Manipulation

- Laser modulator
  - Laser modulator for energy modulation (TEM00 mode laser)

$$\delta = \delta + A\sin(k_L z)$$

  - Laser modulator for angular modulation modulation (TEM01 mode laser, tilted TEM00 laser, etc)

$$y' = y' + A\sin(k_L z)$$
$$\delta = \delta + Ak_L y\cos(k_L z)$$

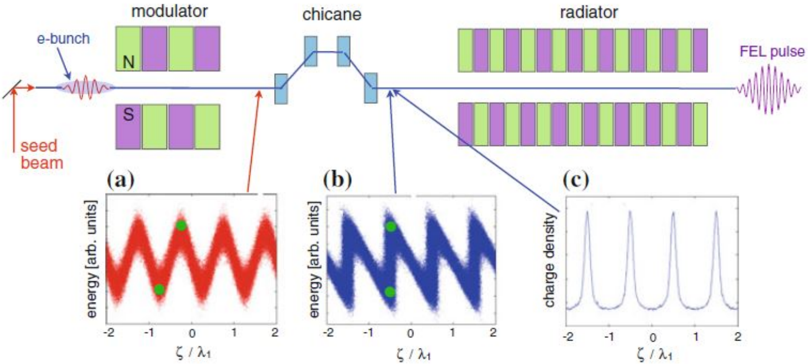
- Magnetic lattice to realized the required transport matrix terms
  - Magnetic chicane to convert energy modulation into density modulation

$$z = z + R_{56}\delta$$

  - Dogleg or dipole: introduce transverse-longitudinal coupling by D or D'
- You can think of/invent more

High-Gain Harmonic Generation (HGHG)

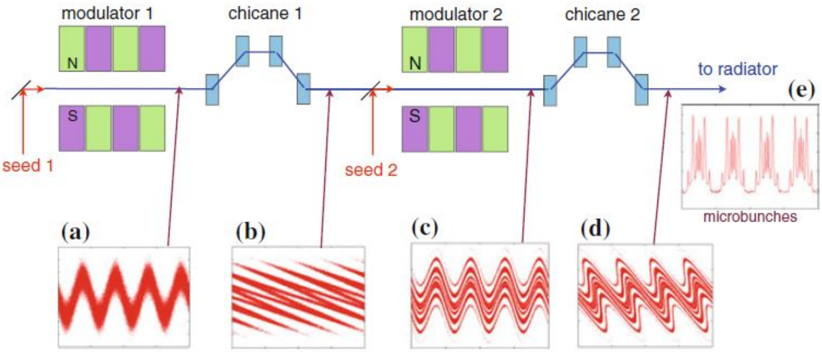
- Laser inducted energy modulation convert to density modulation by a magnetic chicane (L. H. Yu, PRA, 1991 & Science, 2000)



- Frequency up-conversion of the seed laser. Roughly, n-th harmonic bunching requires an energy modulation depth of $n\sigma_\delta$. Single-stage harmonic number limited to about 10.

Echo-Enabled Harmonic Generation (EEHG)

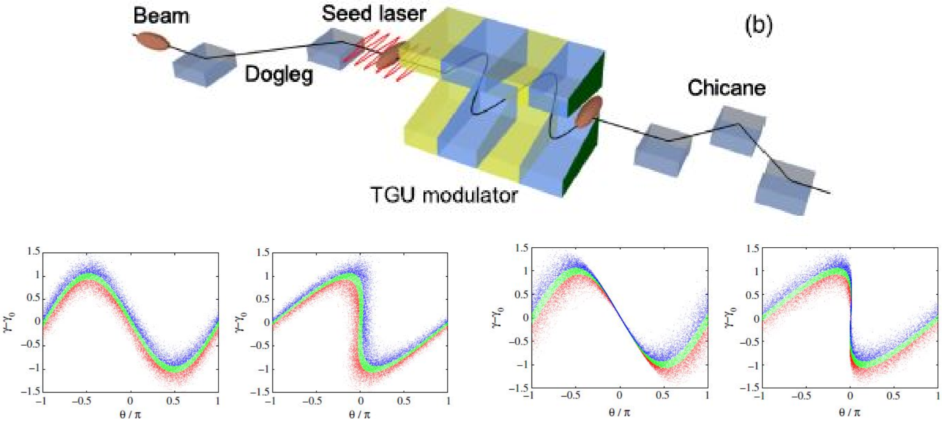
- Two stages of "modulation + dispersion" (G. Stupakov, PRL, 2009)



- Compared to HGHG, the required energy modulation depth is much smaller to realize very high-harmonic bunching. 75-th harmonic bunching realized in experiment.

Phase-merging Enhanced Harmonic Generation (PEHG)

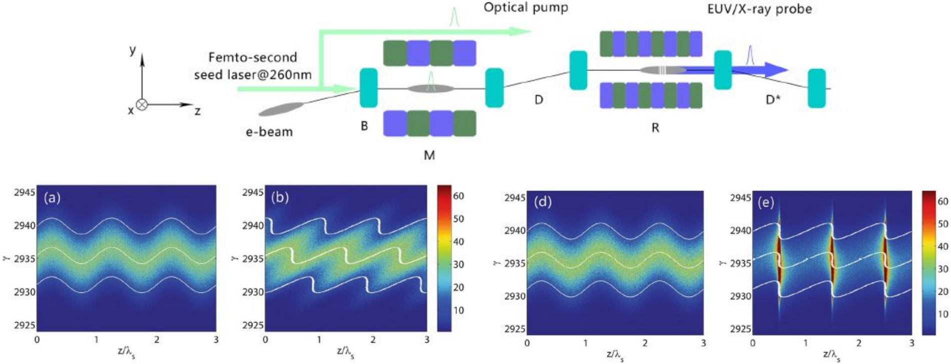
- Transverse-longitudinal coupling (D generated by the dogleg) dynamics is invoked, like a partial transverse-to-longitudinal emittance exchange at the optical laser wavelength range (Deng & Feng, PRL, 2013)



- This technique requires a small transverse emittance.

Angular-dispersion Enhanced Microbunching (ADM)

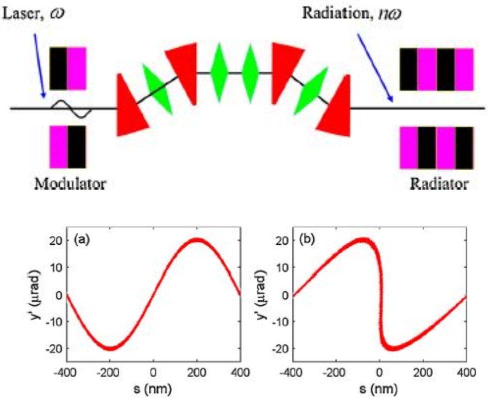
- Spirit similar to that of PEHG, but use dispersion angle D' to introduce the transverse-longitudinal coupling (Feng & Zhao, Scientific Report, 2017)



- A small transverse emittance is required.

Angular Modulation Induced Microbunching

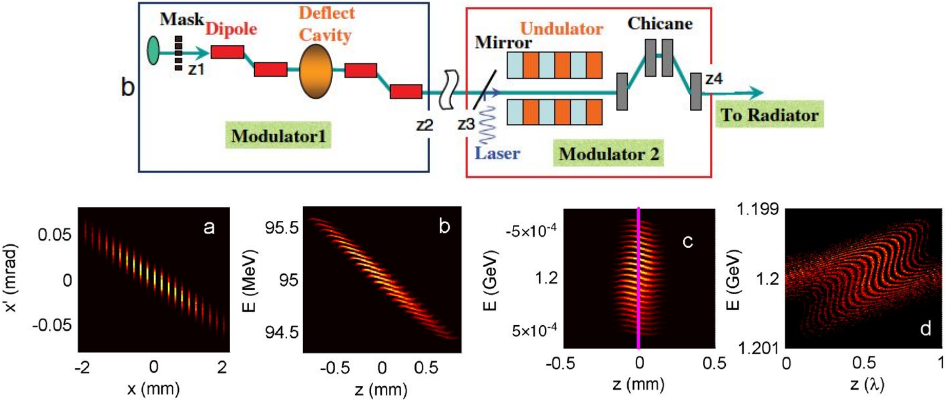
- Similar to HGHG, but replace energy modulation with angular modulation, R_{56} replaced with R_{54} (Xiang & Wan, PRL, 2010)



- This technique also invokes transverse-longitudinal coupling dynamics, and requires a small vertical beam divergence.

Emittance Exchanged-based Microbunching

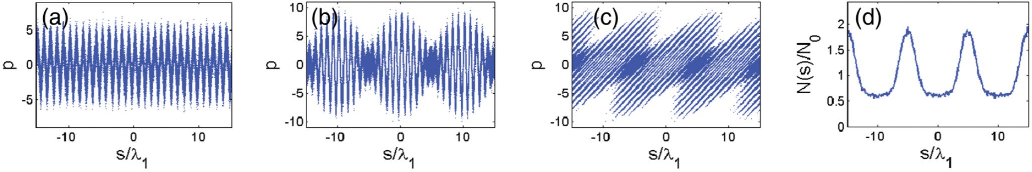
- Longitudinal microbunching can be produced using a transverse mask + transverse-to-longitudinal emittance exchange. And for even higher harmonic generation, one more stage of HGHG can be added (Jiang, et al., PRL, 2010)



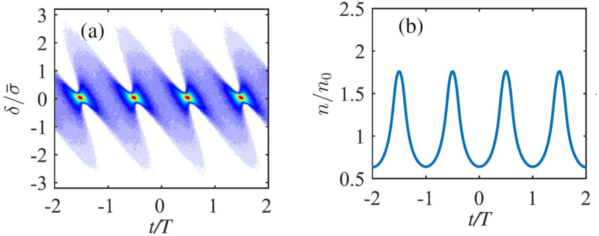
- A special case of transverse-longitudinal coupling. Phase space evolution after emittance exchange shares some similarities with that in EEHG.

Laser Manipulation for Coherent THz Generation

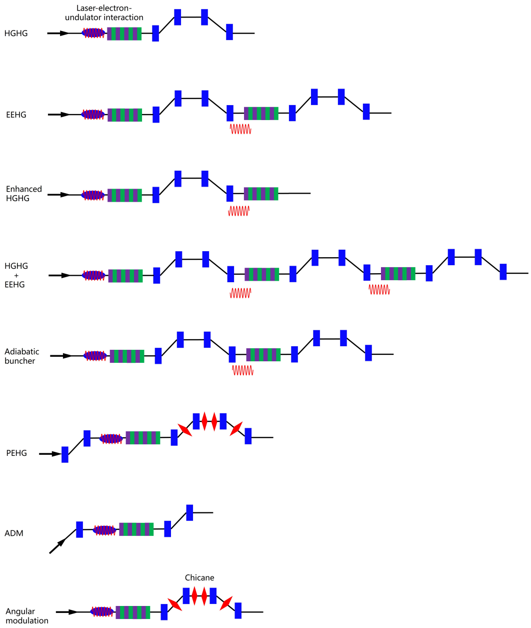
- Beating laser modulation: two lasers with close wavelengths, beat at THz

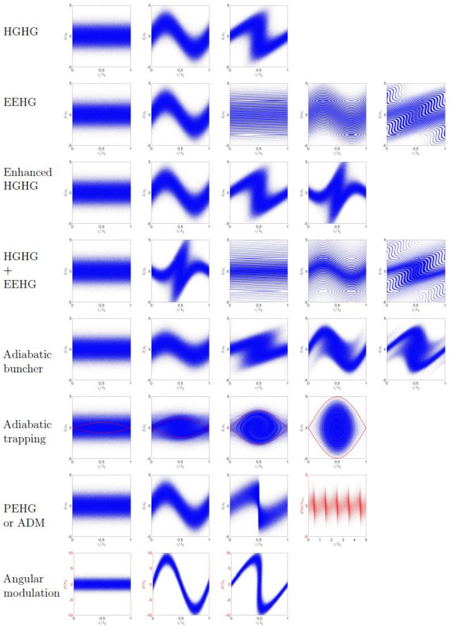


- Laser pulse train modulation: the envelope of pulse train is at the THz wavelength (Zhang, et al., PRAB, 2017)



A Non-exhaustive Summary of Microbunching Techniques





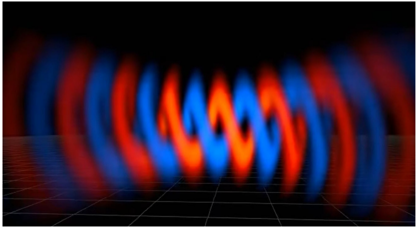
Light with Orbit Angular Momentum

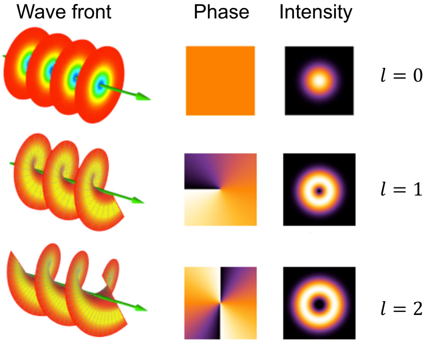
- A Lauguerre-Gaussian laser beam with phase term $e^{-il\phi}$ has well-defined orbit angular momentum (OAM) of $J_z = l\hbar$ per photon (Allen, et al., PRA, 1992), where l is the azimuthal mode index and is called the topological charge

$$u_{pl}(r,\phi,z) = \frac{C}{\sqrt{1+\frac{z^2}{Z_R^2}}}\left(\frac{\sqrt{2}r}{w(z)}\right)^{l} L_p^l\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)\exp\left(-\frac{ikr^2 z}{2\left(z^2+Z_R^2\right)}\right)\exp\left[i(2p+l+1)\tan^{-1}\left(\frac{z}{Z_R}\right)\right]\exp\left(-il\phi\right)$$

$$\mathbf{J} = \epsilon_0 \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d\mathbf{r}$$

- Characteristics: a wavefront shaped as a helix, phase singularity at the center, donut-shaped intensity distribution (necessary but not sufficient, phase is the key)





OAM Light Generation

- Conventional OAM light generation:
  - Spiral phase plate, diffraction mask, mode converter optics

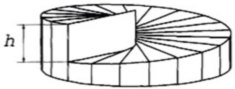
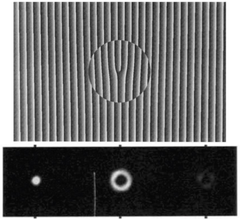


Fig. 1. Sketch of the spiral phaseplate.



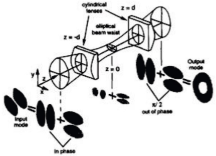


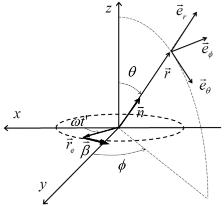
Fig. 4. The cylindrical lens mode converter. If the input $HG_{1,0}$ mode is oriented at 45° with respect to the cylinder axis of lens the mode is converted into the LG_0^1 mode.

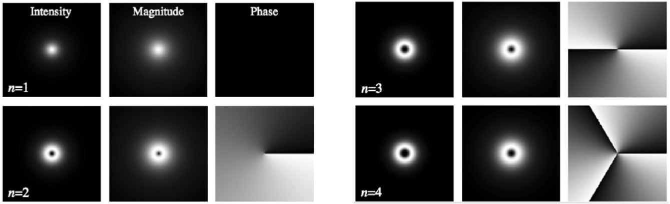
- Particle accelerator-based approaches:
  - Tailor the electron trajectory: electron in helical motion
  - Tailor the electron beam distribution: helical microbunching
  - Radiation acting back on the electron beam: FEL-mode selection based schemes

Radiation of Electron in Helical Motion

- The photons of n-th harmonic carry $n\hbar$ total angular momentum (spin angular momentum + orbit angular momentum) for each

$$
\begin{aligned}
\mathbf{E} &= E_x \mathbf{e}_x + E_y \mathbf{e}_y + E_z \mathbf{e}_z \\
&= \frac{E_x - iE_y}{\sqrt{2}} \mathbf{e}_+ + \frac{E_x + iE_y}{\sqrt{2}} \mathbf{e}_- + E_z \mathbf{e}_z \\
&= \sum_{n=1}^{\infty} \left\{ \frac{i(C_\theta \cos\theta - C_\phi)}{\sqrt{2}} \frac{e^{i\psi_0 + ikR + i(n-1)\phi}}{R} \mathbf{e}_+ + \frac{i(C_\theta \cos\theta + C_\phi)}{\sqrt{2}} \frac{e^{i\psi_0 + ikR + i(n+1)\phi}}{R} \mathbf{e}_- \right. \\
&\quad \left. - iC_\theta \sin\theta \frac{e^{i\psi_0 + ikR + in\phi}}{R} \mathbf{e}_z \right\}
\end{aligned}
$$

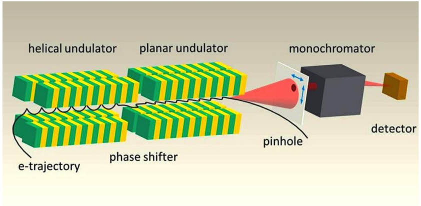


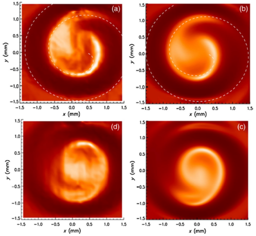


- Astonishing that it is not analyzed before until only recently (Katoh, et al., PRL, 2017). Trust me, you're capable of deriving this also!
- Question: where does the angular momentum of the radiated photon come from?

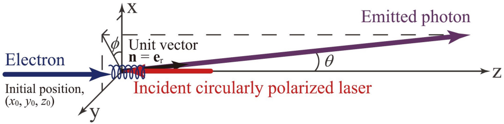
Electron in Helical Motion for OAM Light Generation

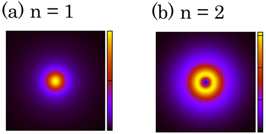
- High-harmonic helical undulator radiation (Sasaki & McNulty, PRL, 2008. Bahrdt, et al., PRL, 2013)





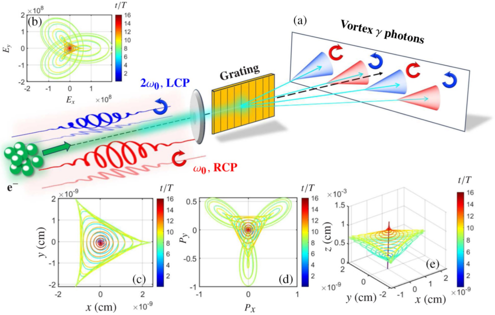
- ICS with a circularly polarized laser (Jentschura & Serbo, PRL, 2011, Taira, et al, Scientific Reports, 2017)



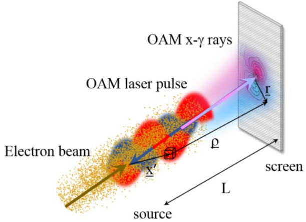


More Sophisticated Helical Motion and Other Ideas

- ICS with two-color circularly polarized lasers (Taira & Katoh, PRL, 2018, Jiang, et al., PRL, 2025), and similarly helical undulator with multiple harmonics



- ICS with an incident laser with OAM (Petrillo, et al. PRL, 2016)



Helical Microbunching for OAM Light Generation

- Can we shape the beam distribution but still generate light with OAM at the fundamental frequency? The answer is yes. To generate OAM light, what we want is to let $E(\omega)$ has a phase term of $e^{-il\phi}$. Then for a long helical electron beam (N_e electrons per λ_L), we have the superimposed radiation field

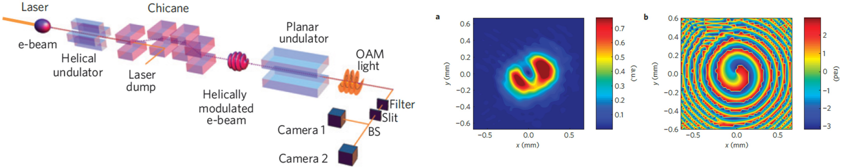
$$x = r\sin(k_L z),\ y = r\cos(k_L z),\ \zeta \equiv k_L z,$$

$$\begin{aligned} E(\omega_L) &= N_e E_{\text{single}}(\omega_L)\frac{1}{2\pi}\int_0^{2\pi} e^{-i(r\sin\theta\cos\phi\sin\zeta + r\sin\theta\sin\phi\cos\zeta+\zeta)}d\zeta \\ &= N_e E_{\text{single}}(\omega_L)\frac{1}{2\pi}\int_0^{2\pi} e^{-i(r\sin\theta\sin(\zeta+\phi)+\zeta)}d\zeta \\ &= N_e E_{\text{single}}(\omega_L)\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} J_n(-r\sin\theta)e^{i(n-1)\zeta+in\phi}d\zeta \\ &= N_e E_{\text{single}}(\omega_L)J_1(-r\sin\theta)e^{i\phi} \end{aligned}$$

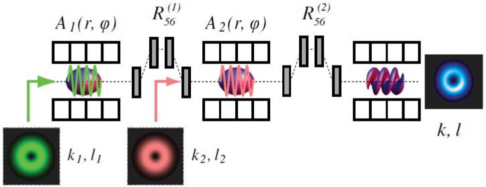
- Key: the particle transverse and longitudinal coordinates should be correlated in a helical pattern. For quantification of this helical microbunching, we can generalize the bunching factor definition to be $b(k,l) = \frac{1}{N_e}\sum_{i=1}^{N_e} e^{-i(kz_i+l\phi_i)}$
- Question: where does the angular momentum of the radiated photons come from?

Helical Microbunching Generation

- General idea: helical energy modulation + dispersion ⇒ helical microbunching
- Modulation method 1: helical undulator + TEM00 laser resonate at high harmonic (Hemsing, et al., PRL, 2009 & Nature Physics, 2013)

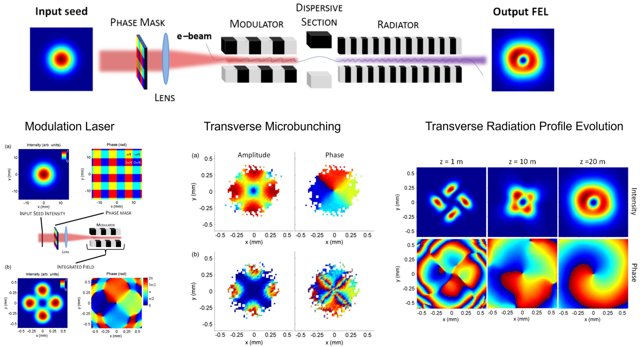


- Modulation method 2: planar undulator + LG laser modulation. For two stages of this, we have echo-enabled helical microbunching which can extend to OAM light generation with shorter wavelength (Hemsing & Marinelli, PRL, 2012)

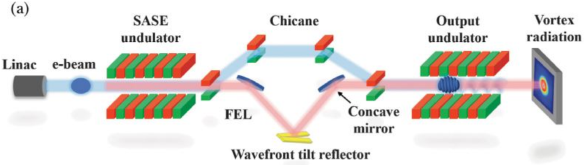


FEL-mode Selection-based OAM Light Generation

- Transverse staircaselike phase pattern for incident laser (Ribič, et al. PRL, 2014). Different transverse part microbunches at different longitudinal position (resemble a helical microbunching structure)

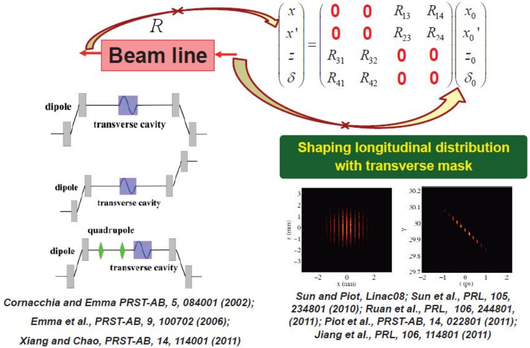


- Wavefront shaping during FEL gain process (Zhou, et al., PRL, 2025)



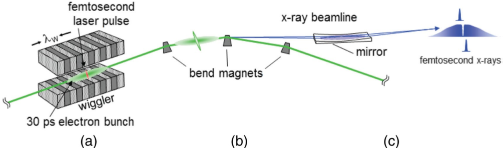
Other Beam Conditioning Techniques (ref. Hemsing, et al., RMP, 2014)

- Emittance exchange (figure from XIANG Dao)



$$\begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 & R_{13} & R_{14} \\ 0 & 0 & R_{23} & R_{24} \\ R_{31} & R_{32} & 0 & 0 \\ R_{41} & R_{42} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ z_0 \\ \delta_0 \end{pmatrix}$$

- Femtoslicing in storage rings, and attosecond pulse generation in FELs (shorten electron-beam lasing part to attosecond range)



Research Directions of Accelerator Light Sources

- Direction 1: Tailor the Radiator Field or Electron Trajectory
  - Bending Magnet as Radiator: Synchrotron Radiation
  - Undulator Magnet as Radiator: Undulator Radiation
  - Laser as Radiator: Inverse Compton Scattering
- Direction 2: Tailor the Electron Beam Distribution
  - Longitudinal Microbunching
  - Helical Microbunching & OAM Light Generation
  - Beam Conditioning
- Direction 3: Radiation Acting Backing on Electron Beam
  - Diffraction-limited Storage Rings
  - Steady-State Micro-Bunching Storage Rings
  - Free-Electron Lasers

Radiation Acting Back on Electron Beam

- Relativistic ($\beta \approx 1$) Larmor formula of radiation power

$$P \propto E_0^2 B_0^2$$

- Overall radiation loss per particle per turn in a storage ring

$$U_0 \propto E_0^3 B_0$$

- Radiation damping: high-energy particle tends to loss more energy
- Quantum excitation: quantum/discrete nature of the emitted photons

$$\langle u \rangle = \frac{8}{15\sqrt{3}} u_c, \ \langle u^2 \rangle = \frac{11}{27} u_c^2, \ \dot{\mathcal{N}} \langle u^2 \rangle = \frac{55}{24\sqrt{3}} \alpha \hbar^2 c^3 \frac{\gamma^7}{|\rho|^3}$$

- Note that it is $\dot{\mathcal{N}} \langle u^2 \rangle$, instead of $\dot{\mathcal{N}} \left(\langle u^2 \rangle - \langle u \rangle^2 \right)$, that determine the quantum excitation of beam emittance. The reason is that the radiation is a Poisson process. Campbell's theorem.

Electron Storage Rings (Deng, et al., NST, 2025)

- Courant-Snyder invariants of particle motion in a storage ring

$$J_I \equiv \frac{(\mathbf{SX})^T \mathbf{T}_I \mathbf{SX}}{2} = \frac{(x - D_x\delta)^2 + \left[\alpha_x (x - D_x\delta) + \beta_x (x' - D'_x\delta)\right]^2}{2\beta_x},$$

$$J_{II} \equiv \frac{(\mathbf{SX})^T \mathbf{T}_{II} \mathbf{SX}}{2} = \frac{(y - D_y\delta)^2 + \left[\alpha_y (y - D_y\delta) + \beta_y (y' - D'_y\delta)\right]^2}{2\beta_y},$$

$$J_{III} \equiv \frac{(\mathbf{SX})^T \mathbf{T}_{III} \mathbf{SX}}{2} = \frac{(z + D'_x x - D_x x' + D'_y y - D_y y')^2 + \left[\alpha_z (z + D'_x x - D_x x' + D'_y y - D_y y') + \beta_z\delta\right]^2}{2\beta_z}.$$

- Equilibrium emittances given by the balance of radiation damping and quantum excitation

$$\epsilon_x \equiv \langle J_I \rangle = \frac{C_L\gamma^5}{2c\alpha_I}\oint \frac{\beta_{55}^{I}}{|\rho|^3} ds = C_q\frac{\gamma^2}{J_x}\frac{I_{5x}}{I_2},\ \epsilon_y \equiv \langle J_{II} \rangle = \frac{C_L\gamma^5}{2c\alpha_{II}}\oint \frac{\beta_{55}^{II}}{|\rho|^3} ds = C_q\frac{\gamma^2}{J_y}\frac{I_{5y}}{I_2},\ \epsilon_z \equiv \langle J_{III} \rangle = \frac{C_L\gamma^5}{2c\alpha_{III}}\oint \frac{\beta_{55}^{III}}{|\rho|^3} ds = C_q\frac{\gamma^2}{J_z}\frac{I_{5z}}{I_2},$$

$$I_2 = \oint \frac{1}{\rho^2} ds,\ I_{4x} = \oint D_x \left(\frac{1-2n}{\rho^3}\right) ds,\ I_{5x} = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds,\ I_{5y} = \oint \frac{\mathcal{H}_y}{|\rho|^3} ds,\ I_{5z} = \oint \frac{\beta_z}{|\rho|^3} ds,$$

$$\mathcal{H}_{x,y} = \beta_{55}^{I,II} = \frac{D_{x,y}^2 + \left(\alpha_{x,y} D_{x,y} + \beta_{x,y} D'_{x,y}\right)^2}{\beta_{x,y}},\ \alpha_I = \frac{U_0}{2E_0} J_x,\ \alpha_{II} = \frac{U_0}{2E_0} J_y,\ \alpha_{III} = \frac{U_0}{2E_0} J_z,\ J_x = 1 - \frac{I_{4x}}{I_2},\ J_y = 1,\ J_z = 2 + \frac{I_{4x}}{I_2}.$$

  - γ: special relativity
  - $\hbar$: quantum mechanics
  - Courant-Snyder optical functions: classical/Hamiltonian mechanics/sympletic matrix analysis/normal form analysis

Theoretical Minimum Horizontal Emittance

- Evolution of $\mathcal{H}_x$ in a bending magnet:

$$\mathcal{H}_x(\alpha) \equiv \beta^I_{55}(\alpha) = 2|\mathbf{E}_{I5}(\alpha)|^2 = 2|\left(\mathbf{B}(\alpha)\mathbf{E}_I(0)\right)_5|^2$$
$$= \left(\sqrt{\beta_{x0}}\left(\sin\alpha + D'_{x0}\right) + \frac{\alpha_{x0}}{\sqrt{\beta_{x0}}}\left[D_{x0} - \rho(1-\cos\alpha)\right]\right)^2 + \left(\frac{D_{x0} - \rho(1-\cos\alpha)}{\sqrt{\beta_{x0}}}\right)^2.$$

- The minimum emittance is realized when in each bending magnet center we have

$$\alpha_{x0} = 0,\ \beta_{x0} \approx \frac{\rho\theta}{2\sqrt{15}},\ D_{x0} \approx \frac{\rho\theta^2}{24},\ D'_{x0} = 0,$$

and the theoretical minimum horizontal emittance is (Teng, 1984)

$$\epsilon_{x,\min} = C_q\frac{\gamma^2}{J_x}\frac{\theta^3}{12\sqrt{15}},\ \epsilon_{x,\min}[\mathrm{nm}] = 31.6E_0^2[\mathrm{GeV}]\theta^3[\mathrm{rad}].$$

with $C_q = 3.8319 \times 10^{-13}$ m for electrons.

Low-Transverse-Emittance Storage Rings

- TME lattice: minimizing the average $\beta_{55}^{I,II,III}$ ($\mathcal{H}_x$) in the bending magnet
- From DBA lattice to MBA lattice: increase the number of bending magnets to decrease θ, thus to lower beam emittance
- Longitudinal gradient bend: more quantum excitation at places where $\beta_{55}^{I,II,III}$ are small, and less excitation where they are large
- Transverse gradient bend: damping partition
- Robinson wigglers: damping partition
- Reverse bend: more radiation damping
- Damping wigglers: more radiation damping

- Refer to the talk of Prof. BAI Zhenghe for more details

Theoretical Minimum Longitudinal Emittance

- Evolution of β_z in a bending magnet:

$$\beta_z(\alpha) \equiv \beta_{55}^{III}(\alpha) = 2|\mathbf{E}_{III5}(\alpha)|^2 = 2|\left(\mathbf{B}(\alpha)\mathbf{E}_{III}(0)\right)_5|^2$$
$$= \left(\sin\alpha \frac{\alpha_{z0}}{\sqrt{\beta_{z0}}} D_{x0} + \rho(1-\cos\alpha)\frac{\alpha_{z0}}{\sqrt{\beta_{z0}}} D'_{x0} + \sqrt{\beta_{z0}} - \rho\left(-\alpha + \sin\alpha\right)\frac{\alpha_{z0}}{\sqrt{\beta_{z0}}}\right)^2$$
$$+ \left(-\sin\alpha \frac{1}{\sqrt{\beta_{z0}}} D_{x0} - \rho(1-\cos\alpha)\frac{1}{\sqrt{\beta_{z0}}} D'_{x0} + \rho\left(-\alpha + \sin\alpha\right)\frac{1}{\sqrt{\beta_{z0}}}\right)^2.$$

- The minimum emittance is realized when in each bending magnet center we have

$$\alpha_{z0} = 0,\ \ \beta_{z0} \approx \frac{\rho\theta^3}{120\sqrt{7}},\ \ D_{x0} \approx -\frac{\rho\theta^2}{40},\ \ D'_{x0} = 0,$$

  and the theoretical minimum longitudinal emittance is (Zhang, et al., PRAB, 2021. Deng, et al., NST, 2025)

$$\epsilon_{z,\min} = C_q \frac{\gamma^2}{J_z} \frac{\theta^3}{60\sqrt{7}}.$$

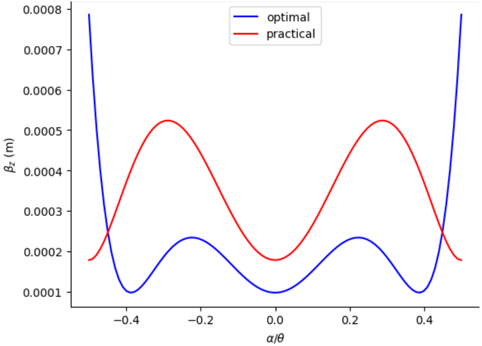
  with $C_q = 3.8319 \times 10^{-13}$ m for electrons.

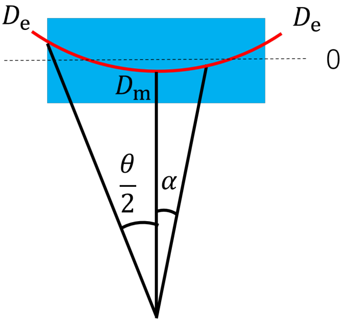
More Practical Minimum Longitudinal Emittance

- It is difficult to make the optimal conditions satisfied in all bending magnets. A more practical way to realize small longitudinal emittance is to letting each half of the bending magnet is isochronous (Deng, et al. NST, 2025)

$$\alpha_{z0} = 0, \;\; \beta_{z0} \approx \frac{\rho\theta^3}{12\sqrt{210}}, \;\; D_{x0} \approx -\frac{\rho\theta^2}{24}, \;\; D'_{x0} = 0, \;\; \epsilon_{z,\min,\mathrm{ISO}} = C_q \frac{\gamma^2}{J_z} \frac{\theta^3}{6\sqrt{210}}.$$

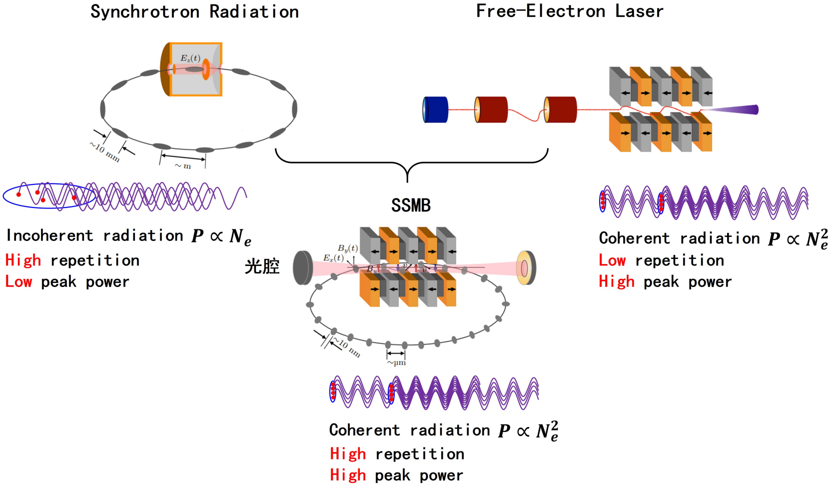
- An example evolution of β_z in the magnet for $\rho = 1$ m, $\theta = \frac{\pi}{10}$ with $\alpha \in [-\frac{\theta}{2}, \frac{\theta}{2}]$ for the two cases is shown below





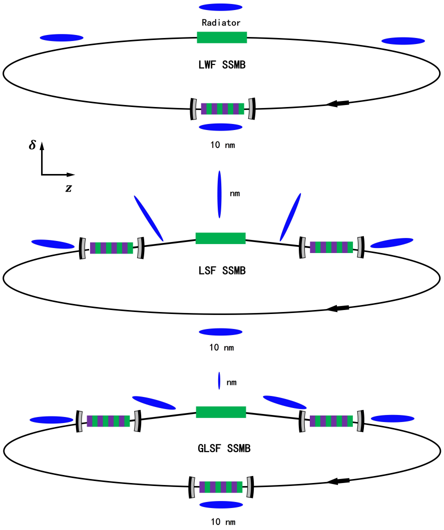
Steady-State Micro-Bunching Storage Rings (Ratner & Chao, PRL, 2010. Tang & Deng, Acta Physica Sinica, 2021)

- Microwave replaced by laser, bunch length shrinked by 6 orders of magnitude
- High-repetition high-power coherent radiation, with wavelength ranging from THz to soft X-ray



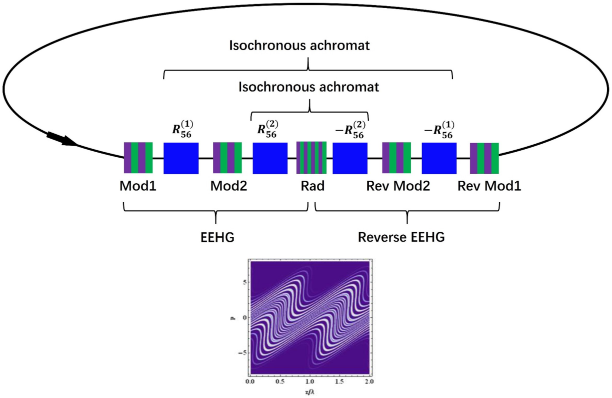
From Longitudinal Weak Focusing to Strong Focusing, to Generalized Longitudinal Strong Focusing (Deng, et al., NST, 2025)

- **Longitudinal weak focusing (LWF)**: limited by the lower limit of reachable phase slippage factor of the ring, the lower limit of the bunch length is tens of nanometers
- **Longitudinal strong focusing (LSF)**: nm microbunch can be realized, but the required modulation laser power is in GW level
- **Generalized longitudinal strong focusing (GLSF, Li, et al., PRAB, 2023)**: transverse-longitudinal coupling dynamics is invoked, ultrasmall vertical emittance in a planar storage ring is exploited



Echo SSMB (Deng, Pan, Zhao, Li, Chao & Tang, 2025, submitted)

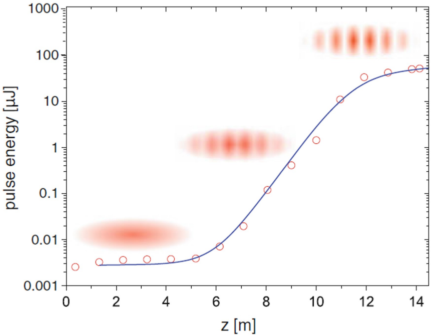
- **Issues of GLSF**: IBS and nonlinear transverse-longitudinal dynamics, both due to the fact that the laser modulators are placed at dispersive locations
- **Solution: Echo SSMB**
  - Laser modulator placed at dispersion-free location, but a microbunching scheme requiring much smaller modulation laser power compared to HGHG in realizing high-harmonic bunching – EEHG
  - To do echo turn-by-turn in a storage ring – reversible modulation



Parameter	Value
Circumference	~ 300 m
Energy	~ 600 MeV
Current	≤ 1 A
Wavelength	5 – 100 nm
13.5 nm power (in 2% b.w.)	> 1 kW
Photon flux (in 1 mev)	> 10^{13} phs/s

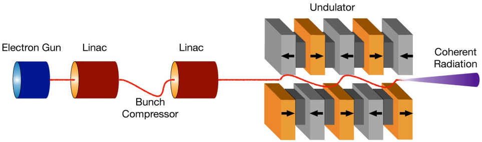
Another Important Example of Radiation Acting Back on the Electron Beam is Free-Electron Lasers (FEL)

- From undulator radiation to free-electron laser



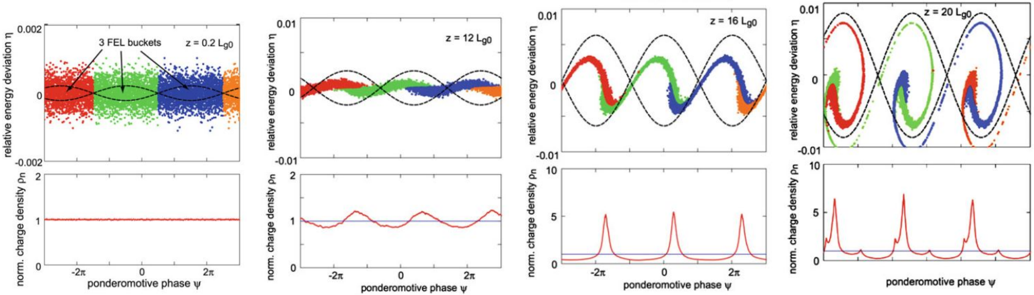
Incoherent $P \propto N_e$

Coherent $P \propto N_e^2$ **Microbunching is the key**



Physical Picture of a High-gain FEL

- Electron-Undulator-Radiation coupled system:
  - laser/radiation electric field drives electron beam energy modulation
  - energy modulation converted to density modulation to form microbunching
  - microbunched beam drives coherent radiation



- Figures from Schmüser, Dohlus, Rossbach & Behrens, Springer, 2014.
- Refer to the slides of Prof. LIU Tao for more details about FEL physics.

Electron and Radiation as a Dynamical System (Chao Textbook, 2020)

- For this purpose, let us add a co-propagating laser inside the undulator. Then they interact in the undulator:

$$\vec{E} = -\hat{x}E_0 \sin(\omega_L t - k_L z + \psi_0)$$
$$\frac{d\mathcal{W}}{dt} = e\vec{v}\cdot\vec{E} = \frac{KceE_0}{\gamma}\sin k_u z \sin(\omega_L t - k_L z + \psi_0)$$
$$= -\frac{KceE_0}{2\gamma}\left[\cos(k_u z + \omega_L t - k_L z + \psi_0) - \cos(k_u z - \omega_L t + k_L z - \psi_0)\right]$$

- In order for the first cosine term to deliver or extract energy systematically from the electron, we need $k_u\bar{v}_z + kc - k\bar{v}_z = 0$ or $\frac{\bar{v}_z}{c} = \frac{k}{k-k_u}$ which is impossible.
- The second term is nonoscillatory only when

$$\boxed{\frac{\bar{v}_z}{c} = \frac{k}{k+k_u} \Rightarrow \lambda_L = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right).}$$

which is exactly the resonance condition for on-axis radiation.

- This is an extremely important result: the condition for sustained energy transfer all along the undulator yields the exactly same light wavelength as is observed in undulator radiation at $\theta = 0$. It is the reason why spontaneous undulator radiation can serve as seed radiation in an FEL.

Ponderomotive Phase

- It is customary in FEL physics to call the argument ψ of the second cosine function the *ponderomotive phase*

$$\boxed{\psi(t) = (k_L + k_u)z(t) - \omega_L t - \psi_0}.$$

So

$$\frac{d\mathcal{W}}{dt} = e\vec{v}\cdot\vec{E} = \frac{KceE_0}{2\gamma}\cos\psi$$

- The ponderomotive phase ψ and electron energy δ are both dynamical variables

$$\psi = (k_u\bar{v}_z - \omega_L + k_L\bar{v}_z)t + \psi_0$$

$$\delta = \frac{\gamma - \gamma_r}{\gamma_r},\ \ |\delta| \ll 1$$

- The rate of change ψ is

$$\frac{d\psi}{dt} \approx k_u\bar{v}_z - \omega_L + k_L\bar{v}_z$$

With $\gamma \approx \gamma_r$, it follows that $\bar{v}_z \approx c\left[1 - \frac{1+K^2/2}{2\gamma^2}(1-2\delta)\right]$, from which we have

$$\frac{d\psi}{dt} \approx 2k_u c\delta.$$

The FEL Pendulum Equations

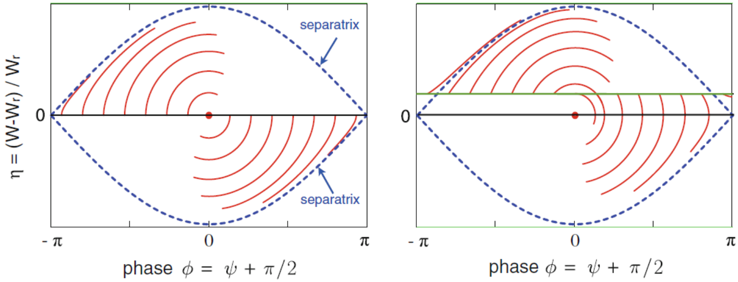
- The rate of change of electron energy (for better comparison with the mathematical pendulum, we have shifted the zero phase definition of ψ)

$$\frac{d\delta}{dt} = \frac{1}{\gamma_r mc^2}\frac{d\mathcal{W}}{dt} = -\frac{KceE_0}{2\gamma_r^2 mc^2}\sin\psi.$$

- Combining what we have so far,

$$\frac{d\psi}{dt} \approx 2k_u c\delta, \quad \frac{d\delta}{dt} = -\epsilon \sin\psi$$

with $\epsilon = \frac{KeE_0}{2\gamma_r^2 mc^2}$.



Maxwell-pendulum Equations

- To describe the laser-electron-undulator system in the high-gain regime, we need to establish a set of coupled Maxwell-pendulum equations.
  - The laser field is driven by the microbunched beam with a varying (exponentiating) bunching factor $\Leftarrow$ Maxwell equations.
  - The increasing laser field continues to microbunch the beam $\Leftarrow$ the pendulum equation.
- To describe the coupled equations, we first introduce three dynamical quantities [Bonifacio/Pellegrini/Narducci, 1984]

$$\begin{aligned} \text{bunching factor } \mathcal{B} &= \langle e^{-i\psi} \rangle \\ \text{energy bunching factor } \mathcal{D} &= \langle \delta e^{-i\psi} \rangle \\ \text{laser field amplitude } \mathcal{E} & \end{aligned}$$

The Maxwell-pendulum equations couple these three quantities in one dynamical system.

Derivation

FEL pendulum equation,

$$\frac{d\psi}{dt} \approx 2k_u c\delta, \;\; \frac{d\delta}{dt} = -\epsilon \sin\psi, \;\; \epsilon = \frac{KeE_0}{2\gamma_r^2 mc^2}$$

$$\begin{aligned}
\frac{d\mathcal{B}}{ds} &= \left\langle -ie^{-i\psi}\frac{d\psi}{ds} \right\rangle \\
&= \langle -ie^{-i\psi} 2k_u\delta \rangle = -2ik_u\mathcal{D} \\
\frac{d\mathcal{D}}{ds} &= \left\langle e^{-i\psi}\frac{d\delta}{ds} - ie^{-i\psi}\frac{d\psi}{ds}\delta \right\rangle \\
&= \langle e^{-i\psi}(-\epsilon\sin\psi) - ie^{-i\psi}2k_u\delta^2 \rangle \\
&\approx \langle e^{-i\psi}(-\epsilon\sin\psi) \rangle \text{ (dropping the second order term} \propto \delta^2) \\
&\approx \frac{-\epsilon}{2i} = i\frac{eK}{2\gamma_r^2 mc^2}[JJ]\mathcal{E}
\end{aligned}$$

We still need a third equation for $\frac{d\mathcal{E}}{ds}$. Physically this change in laser field comes from radiation from the microbunched electron beam.

Derivation Continued

In the one-dimensional approximation the equation for the x component becomes in complex notation

$$\text{Maxwell equation } \left[\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial s}\right)^2\right]\mathcal{E} = \frac{1}{\epsilon_0 c^2}\left\langle\frac{\partial j_x}{\partial t}\right\rangle \approx \frac{-i\omega}{\epsilon_0 c^2}\langle j_x\rangle$$

$$\text{current density } j_x = env_x = -enc\frac{2K}{\gamma}\sin\left(k_u s\right)$$

$$\text{volume density } n = \frac{N}{L_z\Sigma}$$

$$L_z = \text{bunch length, } \ \Sigma = \text{cross section area of electron beam}$$

$$\text{assume solution } \mathcal{E}(s,t) = \mathcal{E}(s)\exp\left[i(k_L s - \omega_L t)\right]$$

$$\left[-2ik_L\mathcal{E}(s)' - \mathcal{E}(s)''\right] = \frac{i\omega}{\epsilon_0 c^2}enc\frac{2K}{\gamma}\langle\sin\psi\rangle$$

The coupled linear equations

Slowly varying amplitude (SVA) approximation

$$\text{keep only slow terms} \;\Rightarrow -2ik\frac{d\mathcal{E}}{ds} \approx \frac{i\omega}{\epsilon_0 c^2} enc\frac{2K}{\gamma}\langle \sin\psi\rangle$$

$$\text{add a factor } [JJ] \;\Rightarrow \frac{d\mathcal{E}}{ds} = -i\frac{eKN}{2\epsilon_0\gamma L_z\Sigma}[JJ]\mathcal{B}$$

We finally obtain the coupled linear equations,

$$\frac{d\mathcal{D}}{ds} = i\frac{eK}{2\gamma_r^2 mc^2}[JJ]\mathcal{E}$$
$$\frac{d\mathcal{B}}{ds} = -2ik_u\mathcal{D}$$
$$\frac{d\mathcal{E}}{ds} = -i\frac{eKN}{2\epsilon_0\gamma L_z\Sigma}[JJ]\mathcal{B}$$

Interpretation:

- laser/radiation electric field drives electron beam energy modulation
- energy modulation converted to density modulation to form microbunching
- microbunched beam drives coherent radiation

The Cubic Equation of a High-gain FEL

- Let any one of the three quantities be represented by X. Equation of motion for X is

$$\frac{d^3X}{ds^3} = (-2ik_u)\left(i\frac{eK}{2\gamma_r^2 mc^2}[JJ]\right)\left(-i\frac{eKN}{2\epsilon_0\gamma L_z\Sigma}[JJ]\right)X$$
$$= -i\frac{\pi r_0 K^2 k_u N}{2\gamma_r^3 L_z\Sigma}[JJ]^2 X$$

  This is the famous cubic equation for a 1-D FEL. [Bonifacio/Pellegrini/Narducci (1984)]
- Define $\frac{\pi r_0 K^2 k_u N}{2\gamma_r^3 L_z\Sigma}[JJ]^2 \equiv \frac{1}{\left(\sqrt{3}L_G\right)^3}$, where L_G is the *gain length* of the laser power.

  The three independent solutions of X are given by

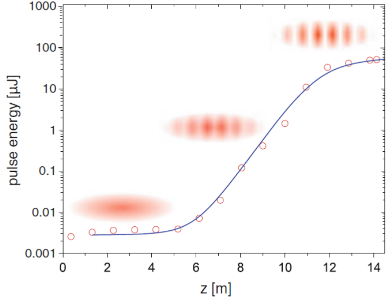
$$X_1 = \exp\left(i\frac{z}{\sqrt{3}L_G}\right) \quad \text{oscillatory solution}$$
$$X_2 = \exp\left(-i\frac{z}{2\sqrt{3}L_G} - \frac{z}{2L_G}\right) \quad \text{damped solution}$$
$$X_3 = \exp\left(-i\frac{z}{2\sqrt{3}L_G} + \frac{z}{2L_G}\right) \quad \text{exponentially growing solution}$$

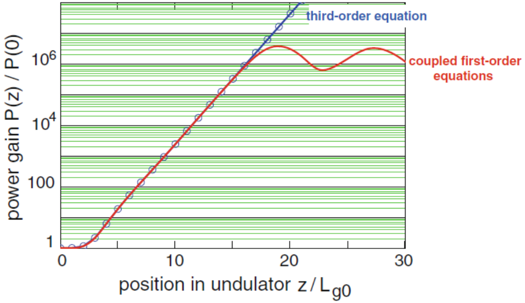
Analytical Solution of the Cubic Equation

- After a beam is injected into the system, starting with initial conditions $\mathcal{B}(0), \mathcal{D}(0), \mathcal{E}(0)$, the system evolves with time. After some distance, all three quantities will be dominated by the solution $X_3 \sim \exp\left(\frac{z}{2L_G}\right)$, which means

$$\text{Laser power} \sim |\mathcal{E}|^2 \sim \exp\left(\frac{z}{L_G}\right)$$

- The exponential growth cannot continue indefinitely. The exponentiation ceases and the laser power saturates when the bunching factor reaches 1.





Seeding

- The exponential growth, however, requires an initial condition to set it off. Had $\mathcal{B}(0), \mathcal{D}(0), \mathcal{E}(0)$ been all zero, there will be no FEL growth.
  - $\mathcal{B}(0) \neq 0 \Rightarrow$ the injected beam is initially microbunched in its longitudinal distribution, even if only slightly.
  - $\mathcal{D}(0) \neq 0 \Rightarrow$ the injected beam is initially microbunched in its energy distribution, even if only slightly.
  - $\mathcal{E}(0) \neq 0 \Rightarrow$ there is an injected laser with the resonant frequency, even if weakly.

- A special case with $\mathcal{B}(0) \neq 0$ is when a nonmicrobunched beam is injected without a laser. A nonzero value of $\mathcal{B}(0)$ comes from a statistical noise in the beam distribution. This amazing possibility, easiest to achieve, yielding a robust facility of brilliant X-ray laser source, is what is currently employed at LCLS. This scheme is called Self-Amplified Spontaneous Emission (SASE).

Summary: Accelerator Light Source is an Versatile and Active Research Ground

- Direction 1: Tailor the Electron Trajectory
  - Bending Magnet as Radiator: Synchrotron Radiation
  - Undulator Magnet as Radiator: Undulator Radiation
  - Laser as Radiator: Inverse Compton Scattering
- Direction 2: Tailor the Electron Beam Distribution
  - Longitudinal Microbunching
  - Helical Microbunching & OAM Light Generation
  - Beam Conditioning
- Direction 3: Radiation Acting Backing on Electron Beam
  - Diffraction-Limited Storage Rings
  - Steady-State Micro-Bunching Storage Rings
  - Free-Electron Lasers

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